Product Formula Approach for Stable FDTD Modeling of Magnetized Plasma

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***Abstract*—A product-formula approach is used to stabilize 3D FDTD in a dispersive anisotropic magnetized plasma medium. The whole FDTD algorithm in a time step is split into three substeps, in two of which the complex medium properties are embedded. Another substep is an unconditionally stable FDTD (US-FDTD) in free space, where a locally one-dimensional (LOD) scheme is used here. The accuracy and stability of the proposed method are verified through simulation of wave propagation in a microwave cavity partially filled with a magnetized plasma slab.**

***Index Terms*— Cavity, finite difference time domain (FDTD), magnetized plasma, unconditionally stable.**

I. Introduction

THE finite difference time domain method has been largely utilized in various electromagnetic problems involving complex media [1]. Specifically, there has been efforts to simulate wave propagation in dispersive and anisotropic media. There are many natural and artificial anisotropic materials that are frequently used in radar cross section control [2], microwave and Terahertz antenna substrates [3,4], a large class of optical components [5] and many other applications.

In the presence of an external applied magnetic field, a plasma exhibits the anisotropic behavior. It is a suitable medium for frequency shifting [6]. Electromagnetic wave propagation in a magnetized plasma is critically important for investigations of space weather hazards, satellite communications, radar, remote sensing and geophysics [7]. Also, there has been applications in efficiency enhancement and spectrum variety of slow wave structures partially filled with plasma [8].

To date, researchers have proposed several ideas in FDTD transient simulation of magnetized plasma in a variety of research areas [2,7,9]. In these works, the Courant-Friedrich-Levy number (CFLN) can not exceed unity and hence the methods are conditionally stable. However, in applications with fine geometric details and high quality factor, a CFL limit is much restrictive [1]. Although an ADI scheme is introduced to isotropic plasma [10], up to the authors' knowledge no US-FDTD is reported for an anisotropic plasma.

In this paper, we develop a 3-D US-FDTD in magnetized plasma. First, the FDTD algorithm is derived and is proved to be stable. Then, the application of the proposed algorithm is illustrated by computing the field waveforms and resonance frequencies in a PEC cavity partially filled with a magnetized plasma slab. The accuracy of the proposed algorithm is verified by comparing the method with conventional FDTD [2].

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II. Proposed Unconditionally Stable FDTD

Maxwell equations in a dispersive magnetized plasma medium are in the following matrix form

$$\frac{\partial}{\partial t}\begin{bmatrix}\vec{H}\\ \vec{E}\\ \vec{J}\end{bmatrix}=\begin{bmatrix}0 & -\frac{1}{\mu_0}\nabla\times & 0\\ \frac{1}{\varepsilon_0}\nabla\times & 0 & -\frac{1}{\varepsilon_0}\\ 0 & \varepsilon_0\bar{\bar{\omega}}_p^2 & -\bar{\bar{v}}+\vec{\omega}_b\times\end{bmatrix}\begin{bmatrix}\vec{H}\\ \vec{E}\\ \vec{J}\end{bmatrix}, \tag{1}$$

where $\vec{J}$ is the polarization current density, v is the electron collision frequency and represents losses, ω_p is the plasma frequency, and $\vec{\omega}_b=\omega_{bx}\hat{x}+\omega_{by}\hat{y}+\omega_{bz}\hat{z}$ is the electron gyrofrequency. Also, $\bar{\bar{\omega}}_p=\mathrm{diag}(\omega_p)$ and

$$(-\bar{\bar{v}}+\vec{\omega}_b\times)=\begin{bmatrix}-v & -\omega_{bz} & \omega_{by}\\ \omega_{bz} & -v & -\omega_{bx}\\ -\omega_{by} & \omega_{bx} & -v\end{bmatrix}. \tag{2}$$

It is clear that in (1) each element is a 3×3 matrix. To stabilize the algorithm we should transform the matrix in (1) to a skew-symmetric matrix (plus a diagonal matrix with nonpositive diagonal elements when the medium is lossy) [1]. A change of variables fulfills this requirement:

$$\begin{bmatrix}\vec{\tilde{E}}\\ \vec{\tilde{J}}\end{bmatrix}=\begin{bmatrix}\sqrt{\frac{\varepsilon_0}{\mu_0}} & 0\\ 0 & \frac{1}{\omega_p\sqrt{\varepsilon_0\mu_0}}\end{bmatrix}\begin{bmatrix}\vec{E}\\ \vec{J}\end{bmatrix}. \tag{3}$$

Therefore,

$$\frac{\partial}{\partial t}\begin{bmatrix}\vec{H}\\ \vec{\tilde{E}}\\ \vec{\tilde{J}}\end{bmatrix}=\begin{bmatrix}0 & -c_0\nabla\times & 0\\ c_0\nabla\times & 0 & -\bar{\bar{\omega}}_p\\ 0 & \bar{\bar{\omega}}_p & -\bar{\bar{v}}+\vec{\omega}_b\times\end{bmatrix}\begin{bmatrix}\vec{H}\\ \vec{\tilde{E}}\\ \vec{\tilde{J}}\end{bmatrix}, \tag{4}$$

where $c_0=\frac{1}{\sqrt{\varepsilon_0\mu_0}}$. If the matrix in (4) is represented by L and the field vectors by $u=\left[\vec{H},\vec{\tilde{E}},\vec{\tilde{J}}\right]^T$, the fields are updated as follows:

$$u(t+\Delta t)=e^{L\Delta t}u(t). \tag{5}$$

It is easy to show that $\|e^{L\Delta t}\|\leq e^{\rho(L)\Delta t}$, where $\rho(L)$ is the largest eigenvalue of the symmetric matrix $(L+L^T)/2$ [1]. It has six zero eigenvalues and three negative eigenvalues $\lambda=-\upsilon$. Hence, $\|e^{L\Delta t}\|\leq 1$ and the proposed algorithm is stable.

To compute $e^{L\Delta t}$, the decomposition $L=L_0+L_m$ is defined where $L_0=\begin{bmatrix}A & 0\\ 0 & 0\end{bmatrix}$, $L_m=\begin{bmatrix}0 & 0\\ 0 & B\end{bmatrix}$ and

$$A=\begin{bmatrix}0 & -c_0\nabla\times\\ c_0\nabla\times & 0\end{bmatrix}, B=\begin{bmatrix}0 & -\bar{\bar{\omega}}_p\\ \bar{\bar{\omega}}_p & -\bar{\bar{\upsilon}}+\vec{\omega}_b\times\end{bmatrix}. \tag{6}$$

Magnetized plasma material properties are totally embedded in L_m. The matrix L_0 is the free space propagator which includes all spatial derivatives. The update procedure is written as the following product formula [1,11]

$$u(t+\Delta t)\approx e^{L_m\Delta t/2}e^{L_0\Delta t}e^{L_m\Delta t/2}u(t). \tag{7}$$

The approximation error in (7) is always less than $C(\Delta t)^3$ where C is a positive constant [11]. It is seen that each time step is split into three substeps. Using Yee's notation, (7) can be written in the more appropriate form of

$$\begin{bmatrix}\vec{\tilde{E}}_{m1}\\ \vec{\tilde{J}}_m\end{bmatrix}=e^{B\Delta t/2}\begin{bmatrix}\vec{\tilde{E}}^n\\ \vec{\tilde{J}}^n\end{bmatrix}, \tag{8a}$$

$$\begin{bmatrix}\vec{H}^{n+1}\\ \vec{\tilde{E}}_{m2}\end{bmatrix}=e^{A\Delta t}\begin{bmatrix}\vec{H}^n\\ \vec{\tilde{E}}_{m1}\end{bmatrix}, \tag{8b}$$

$$\begin{bmatrix}\vec{\tilde{E}}^{n+1}\\ \vec{\tilde{J}}^{n+1}\end{bmatrix}=e^{B\Delta t/2}\begin{bmatrix}\vec{\tilde{E}}_{m2}\\ \vec{\tilde{J}}_m\end{bmatrix}. \tag{8c}$$

At this stage, we may calculate $e^{B\Delta t/2}$ and $e^{A\Delta t}$. The eigenvalues of B are

$$\lambda_{1,2}=\tfrac{1}{2}\left[-\upsilon\pm\sqrt{\upsilon^2-4\omega_p^2}\right], \tag{9a}$$

$$\lambda_{3,4}=\frac{1}{2}\left[-(\upsilon-j\omega_b)\pm\sqrt{(\upsilon-j\omega_b)^2-4\omega_p^2}\right], \tag{9b}$$

$$\lambda_{5,6}=\frac{1}{2}\left[-(\upsilon+j\omega_b)\pm\sqrt{(\upsilon+j\omega_b)^2-4\omega_p^2}\right]. \tag{9c}$$

Then, according to Cayley-Hamilton theorem

$$e^{B\Delta t/2}=\sum_{i=1}^{6}\omega_i B^{i-1}, \tag{10}$$

in which, ω_i are obtained as follows

$$\Omega=\Lambda^{-1}E, \tag{11a}$$

$$\Omega(i)=\omega_i, \Lambda(i,j)=\lambda_i^{j-1}, E(j)=e^{\lambda_j\Delta t/2}, 1\leq i,j\leq 6. \tag{11b}$$

Hence, there exists a closed-form formula for $e^{B\Delta t/2}$. However, it is lengthy and there is no need to write it down here. The important point is that computation of $e^{B\Delta t/2}$ is performed just once as a pre-processing task and therefore (10) adds almost no computational burden to the problem.

The matrix equation (8a) consists of six equations. For example, the first equation is

$$\tilde{E}_{m1x}=\beta_{11}\tilde{E}_x^n+\beta_{12}\tilde{E}_y^n+\beta_{13}\tilde{E}_z^n+\beta_{14}\tilde{J}_x^n+\beta_{15}\tilde{J}_y^n+\beta_{16}\tilde{J}_z^n, \tag{12}$$

where β_{ij} is the (i,j) element of $e^{B\Delta t/2}$. This is also the case for (8c).

The corresponding components of $\vec{E}$ and $\vec{J}$ vectors are collocated in space in a Yee cell. This means that they are placed in the grid as $(E_x/J_x)(i+1/2,j,k)$, $(E_y/J_y)(i,j+1/2,k)$ and $(E_z/J_z)(i,j,k+1/2)$. With this assumption, there are some components of $\vec{E}/\vec{J}$ on the right-hand side of (8a) and (8c), not collocated in space with the component on the left-hand side. Hence, an averaging scheme from four neighboring points is needed; for example when computing $(E_x/J_x)(i+1/2,j,k)$, other components are averaged as follows:

$$\begin{aligned}(E_y/J_y)(i+1/2,j,k)=\frac{1}{4}[&(E_y/J_y)(i,j-1/2,k)\\ &+(E_y/J_y)(i,j+1/2,k)+(E_y/J_y)(i+1,j-1/2,k)\\ &+(E_y/J_y)(i+1,j+1/2,k)],\end{aligned} \tag{13a}$$

$$\begin{aligned}(E_z/J_z)(i+1/2,j,k)=\frac{1}{4}[&(E_z/J_z)(i,j,k-1/2)\\ &+(E_z/J_z)(i,j,k+1/2)+(E_z/J_z)(i+1,j,k-1/2)\\ &+(E_z/J_z)(i+1,j,k+1/2)],\end{aligned} \tag{13b}$$

There are several ways to compute $e^{A\Delta t}$ which represents the free space propagation. Here we use an LOD-FDTD algorithm. To reduce the number of arithmetic operations the LOD1 fundamental scheme proposed in [12] may be used.

III. Verification

Fig. 1 shows a PEC rectangular cavity partially filled with a magnetized plasma slab. The cavity dimensions are $(60\times 60\times 60)(\Delta x)^3$, where $\Delta x=1$ cm is a FDTD spatial step. The plasma slab is located in the center of the cavity and is $12\Delta x$ thick in the y direction. In the slab considered, $\omega_p=500\times 10^6$ rad/s, $\omega_{bx}=\omega_{by}=\omega_{bz}=10^9$ rad/s and $\upsilon=300$ MHz. As shown in Fig. 1, the source (S) and observation (O) points are located respectively at $(50,10,10)\Delta x$ and $(10,50,50)\Delta x$ on the opposite sides of the slab. The field component H_z at point S is excited by a Gaussian pulse. In [2] a conditionally stable FDTD in magnetized plasma is proposed which is referred to as conventional FDTD in the simulations which takes the role of a benchmark and is always run for CFLN = 1.

Fig. 2(a) shows H_z observed at O as a function of time for conventional- and US-FDTD (CFLN = 10). As shown, the two methods agree well. Fig. 2(b) shows the cavity frequency spectrum at point O. The abscissa is normalized to ω_0 which is the dominant mode resonance frequency of an empty cavity with the same dimensions. As expected, there are some resonances in the cavity in the frequency range shown. Also, the agreement between the two methods verifies the accuracy of US-FDTD.

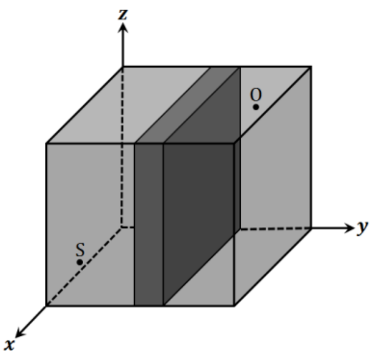
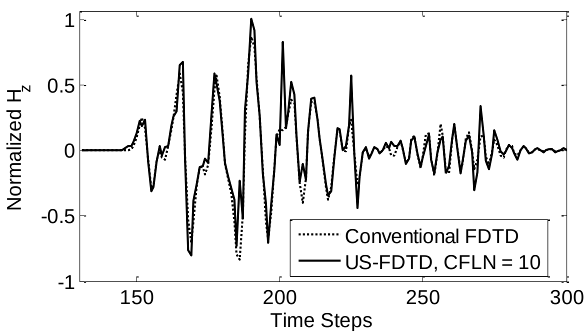
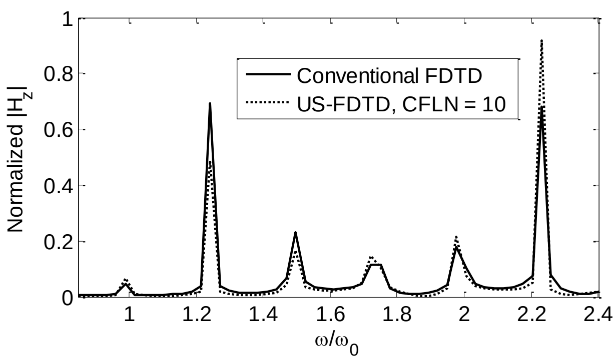


Fig. 1. A PEC cavity partially filled with a plasma slab.



(a)



(b)

Fig. 2. Normalized H_z at point O for FDTD and US-FDTD (CFLN = 10), (a) time-domain waveform, (b) frequency spectrum.

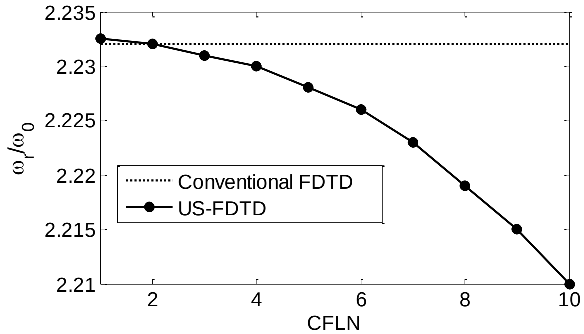


Fig. 3. The cavity resonance frequency for conventional- and US-FDTD (CFLN = 1 to 10).

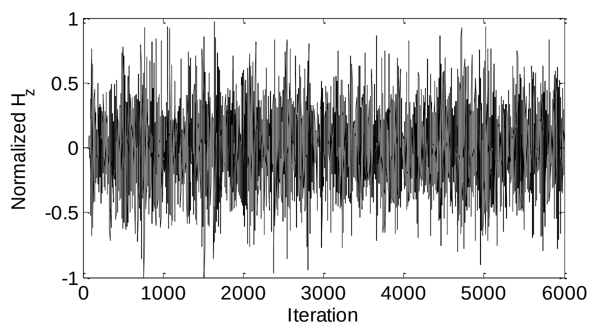


Fig. 4. Normalized H_z computed by US-FDTD at 6000 iterations for CFLN = 100.

Fig. 3 shows one of the resonant frequencies computed by the conventional- and US-FDTD (CFLN = 1 to 10). There is a compromise between the accuracy and computational efficiency. However, the error is small and is 0.9% for CFLN = 10. Finally, Fig. 4 shows normalized H_z computed by US-FDTD for CFLN = 100 in 6000 iterations which clearly indicates the stability of the method.

IV. Conclusion

An unconditionally stable FDTD method using product formula is proposed in dispersive magnetized plasma, which is an anisotropic gyrotropic medium. The method is based on operator splitting and decomposes the problem into free space propagation and medium properties. The method is shown to be accurate and stable through simulation of a cavity partially filled with plasma.

References

[1] A. Taflove, and S. C. Hagness, *Computational Electrodynamics: The Finite Difference Time Domain Method*, Norwell, MA, USA: Artech House, ch. 9 and 18, 2005.

[2] L. Xu, and Naichang Yuan, “FDTD formulations for scattering from 3-D anisotropic magnetized plasma objects,” *IEEE Antennas and Wireless Propagat. Letters*, vol. 5, pp. 335-338, 2006.

[3] O. Madani, M. L. Tounsi, and M. C. E. Yagoub, “Full-wave mode analysis of shielded microstrip discontinuities on anisotropic substrates,” *IEEE Electronics, Circuits and Systems (ICECS)*, pp. 709-712, 2012.

[4] Y. Guo, T. Zhang, W. Y. Yin, and X. H. Wang, “Improved hybrid FDTD method for studying tunable graphene frequency-selective surfaces (GFSS) for THz-wave applications,” *IEEE Trans. On Terahertz Science and Techno.*, vol. 5, pp. 358-367, 2015.

[5] B. Salski, M. Celuch, and W. Guarek, “FDTD for nanoscale and optical problems,” *IEEE Microwave Magazine*, vol. 11, pp. 50-59, 2010.

[6] D. K. Kalluri, S. Eker, and M. M. Ehsan, “Frequency transformation of a whistler wave by a collapsing plasma medium in a cavity: FDTD solution,” *IEEE Trans. Antennas Propagat.*, vol. 57, pp. 1921-1930, 2009.

[7] A. A. Samimi, and J. J. Simpson, “An efficient 3-D FDTD model of electromagnetic wave propagation in magnetized plasma,” *IEEE Trans Antennas Propagat.*, vol. 63, pp. 269-279, 2015.

[8] J. L. Shohet, “Exact solution for the eigenfrequencies of a microwave cavity partially filled with a magnetized plasma,” *J. Appl. Phys.*, vol. 37, pp. 3775-3778, 1966.

[9] J. H. Lee, and D. K. Kalluri, “Three-dimensional FDTD simulation of electromagnetic wave transformation in a dynamic inhomogeneous magnetized plasma,” *IEEE Trans. Antennas Propagat.*, vol. 47, pp. 1146-1151, 1999.

[10] L. Xu, and N. Yuan, “PLJERC-ADI-FDTD method for isotropic plasma,” *IEEE Microw. Wireless Compon. Lett.*, vol. 15, pp. 277-279, 2005.

[11] M. Suzuki, “Decomposition formulas for exponential operators and Lie exponentials with some applications to quantum mechanics and statistical physics,” *J. Math. Phys.*, vol. 26, pp. 601-612, 1985.

[12] E. L. Tan, “Fundamental schemes for efficient unconditionally stable implicit finite-difference time-domain methods,” *IEEE Trans. Antennas Propagat.*, vol. 56, pp. 170-177, 2008.