Diversity of Spectrum Sensing over Frequency Selective Rayleigh Fading Channels

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***Abstract*—In cognitive radio systems, diversity of missed detection probability reflects the reliability of spectrum sensing. In this paper, we quantify the diversity order of missed detection probability over a Rayleigh frequency-selective channel with independent taps. We show that the diversity order is equal to the rank of the primary user's code matrix. If the secondary user gets enough samples from the primary user's signal, the spectrum sensing performance can be improved due to multipath diversity. Also our simulations approve the analytical results.**

***Index Terms*—Cognitive radio, spectrum Sensing, diversity, energy detection, frequency-selective channel.**

I. Introduction

The demand for the wireless radio spectrum is growing rapidly. Meanwhile, a great portion of the licensed spectrum is severely underutilized according to the Federal Communications Commission (FCC) research [1]. Hence, the idea of cognitive radio has emerged as a solution to utilize the spectrum more efficiently by allowing the unlicensed users (secondary users) to access the spectrum when the licensed user (primary user) is idle [2], [3]. For this capability, secondary users first perform spectrum sensing, i.e., they attempt to detect a spectrum hole and then perform data transmission over the detected free band.

Different detection techniques have been proposed for spectrum sensing in the literature. Among them, energy detection is the most common method as it is easy to implement and does not require prior information of the primary user signal [4]. Sensing performance is evaluated by the probabilities of detection and false alarm. Missed detection leads to interference between the primary user and the secondary user and as a result, the performance of data transmission becomes very poor. So having a low missed detection probability is necessary in cognitive radio.

In wireless fading channels, diversity is a fundamental performance indicator and quantifies the effects of independent fading in space, time or frequency. It is defined in terms of the signal-to-noise ratio (SNR) behaviour of the outage probability or the bit error rate for symbol detection [5]. This concept was recently extended to the SNR-dependent behaviour of missed detection probability [6]. Diversity of the missed detection represents the reliability of the spectrum sensing. If the diversity order increases, the probability of missed detection decreases significantly which leads to a more reliable data transmission in a cognitive radio system.

In [7], the average detection probability over a flat fading channel has been analysed. Also, the diversity order of missed detection probability over a flat fading channel has been derived in [6]. But the channel might be frequency-selective, i.e., the signal of the primary user is received at the secondary user through different channel paths. If the same information passes through different independent fading channels, the error probability has a diversity order equal to the number of paths known as multipath diversity [5]. It seems that the average missed detection probability over frequency-selective channel can benefit from multipath diversity.

Several papers have studied the performance of the energy detector over the frequency-selective channel by the central limit theorem approximation which is valid for large number of samples [8], [9], [10]. But the exact analysis and the diversity order of the average missed detection probability over Rayleigh frequency-selective channel has not been analysed.

In this paper by exact analysis of the performance of the energy detector, we show that diversity order of the average missed detection probability over Rayleigh frequency-selective channel is equal to the rank of the primary user's code matrix. We show that by getting enough samples from the primary user's signal, multipath diversity can be achieved.

The remainder of the paper is organized as follows. In section II we describe the system model. The performance of spectrum sensing and the diversity order of the missed detection probability is analysed in section III. We explore the validity of theoretical results by simulation in section IV. Finally, we give our conclusion in section V.

Notation: Matrices are denoted by upper bold face letters and vectors by lower bold face letters. Superscripts ^{H} and ^{T} denote the Hermitian and the transpose of a vector or a matrix. $\Pr(\cdot)$ stands for the probability.

II. System Model

Consider a cognitive network in which a secondary user senses a frequency band to detect the presence or the absence of a primary user. We assume that the channel between the primary user and the secondary user is frequency-selective. So the signal received by the secondary user under the following two hypotheses $\mathcal{H}_0$ and $\mathcal{H}_1$ can be written as

$$\begin{cases} \mathcal{H}_0 : r(n) = \ w(n), & n = 1, ..., N, \\ \mathcal{H}_1 : r(n) = \sqrt{P} \sum\limits_{l=1}^{L} h_l x_p(n-l) + w(n), & n = 1, ..., N, \end{cases} \tag{1}$$

where $x_p(n)$ is the nth normalized sample of the primary user signal, $w(n)$ is the additive white complex Gaussian noise with zero mean and variance σ^2, n represents the sample index and N is the total number of complex samples. The frequency-selective channel is modelled with finite number of taps L and h_l denotes the lth channel filter tap. The tap gains are assumed to be independent, identically distributed Rayleigh with equal unit variance and do not vary over these N symbol times. The primary user transmit power is P. So we define the SNR as $\rho = P/\sigma^2$. $\mathcal{H}_0$ denotes the primary user is absent, while $\mathcal{H}_1$ represents the presence of the primary user.

The above equation can be represented in matrix form as

$$\mathbf{r} = \begin{cases} \sqrt{P}\mathbf{X}\mathbf{h} + \mathbf{w}, & \mathcal{H}_1 \\ \mathbf{w}, & \mathcal{H}_0 \end{cases} \tag{2}$$

where $\mathbf{r} = (r(1), \ldots, r(N))^T$, $\mathbf{h} = (h_1, ..., h_L)^T$, $\mathbf{w} = (w(1), ...w(N))^T$ and the N by L matrix $\mathbf{X}$ which corresponds to the primary user's code matrix is given by

$$\mathbf{X} = \begin{bmatrix} x_p(0) & x_p(-1) & \cdots & x_p(1-L) \\ x_p(1) & x_p(0) & \cdots & x_p(2-L) \\ \vdots & \vdots & \ddots & \vdots \\ x_p(N-1) & x_p(N-2) & \cdots & x_p(N-L) \end{bmatrix}. \tag{3}$$

Under $\mathcal{H}_0$, assuming that the noise samples are independent, $\mathbf{r}$ has a multivariate complex normal distribution with zero mean and covariance matrix $\sigma^2\mathbf{I}$, where $\mathbf{I}$ is the identity matrix of order $N \times N$. Under $\mathcal{H}_1$, for a Given $\mathbf{h}$ and $\mathbf{X}$, $\mathbf{r}$ has a multivariate normal distribution with mean $\boldsymbol{\mu} = \sqrt{P}\mathbf{X}\mathbf{h}$ and covariance matrix $\sigma^2\mathbf{I}$ [11].

III. Spectrum Sensing Performance Analysis

We assume that the secondary user performs energy detection to detect the presence or the absence of the the primary user. So the test statistics of the energy detector at the secondary user can be written as

$$y = \frac{\mathbf{r}^H\mathbf{r}}{\sigma^2} \underset{\mathcal{H}_0}{\overset{\mathcal{H}_1}{\gtrless}} \eta, \tag{4}$$

where η is the decision threshold which satisfies a predetermined false alarm probability. Under $\mathcal{H}_1$, for a given $\mathbf{h}$, the random variable y is non-central Chi-squared distributed with $2N$ degrees of freedom and the non-centrality parameter $\frac{\boldsymbol{\mu}^H\boldsymbol{\mu}}{\sigma^2} = \frac{P\mathbf{h}^H\mathbf{X}^H\mathbf{X}\mathbf{h}}{\sigma^2}$. Likewise, given $\mathcal{H}_0$, y will be central Chi-squared distributed with $2N$ degrees of freedom [11]. The probability of false alarm is defined as $P_f = \Pr(y > \eta|\mathcal{H}_0)$ and is given by [7]

$$P_f = \frac{\Gamma(N, \eta)}{\Gamma(N)}, \tag{5}$$

where $\Gamma(.)$ and $\Gamma(.,.)$ are the the gamma function and the incomplete gamma function, respectively [12].

Conditioned on a specific realization of the channel $\mathbf{h}$, the detection probability which is defined as $P_{d|\mathbf{h}} = \Pr(y > \eta|\mathcal{H}_1)$, can be evaluated by taking the same approach as [7]. Thus, we obtain

$$P_{d|\mathbf{h}} = Q_N\left(\sqrt{2\rho\, \mathbf{h}^H\mathbf{X}^H\mathbf{X}\mathbf{h}}\,, \sqrt{2\eta}\right), \tag{6}$$

where $Q_N(.,.)$ is the generalized Marcum Q-function [12].

We should average $P_{d|\mathbf{h}}$ over the frequency-selective channel $\mathbf{h}$. To this end, We consider the upper and the lower bound of the detection probability.

The matrix $\mathbf{X}^H\mathbf{X}$ is Hermitian and is thus diagonalizable by a unitary transformation, i.e., we can write

$$\mathbf{X}^H\mathbf{X} = \mathbf{U}^H\boldsymbol{\Lambda}\mathbf{U}, \tag{7}$$

where Λ is a diagonal matrix containing the eigenvalues $\lambda_1^2, ..., \lambda_L^2$ of $\mathbf{X^HX}$ and the rows of unitary matrix $\mathbf{U}$ are the corresponding eigenvectors. Note that $\lambda_1, ..., \lambda_L$ are the singular values of the matrix $\mathbf{X}$. Thus, we can write

$$\mathbf{h}^H\mathbf{X}^H\mathbf{X}\mathbf{h} = \sum_{l=1}^{L} \lambda_l^2 |\tilde{h}_l|^2, \tag{8}$$

where $\tilde{\mathbf{h}} = \mathbf{U}\mathbf{h}$ has the same distribution as $\mathbf{h}$. Since all the λ_l^2 are non-negative, by sorting the eigenvalues in a way that the first M ones are non-zero denoted by $\lambda_1^2, ..., \lambda_M^2$, we can obtain the following inequality

$$\lambda_{\min}^2 \sum_{m=1}^{M} |\tilde{h}_m|^2 \le \sum_{l=1}^{L} \lambda_l^2 |\tilde{h}_l|^2 \le \lambda_{\max}^2 \sum_{m=1}^{M} |\tilde{h}_m|^2. \tag{9}$$

By defining $z = \sum\limits_{m=1}^{M} |\tilde{h}_m|^2$ and using the fact that the Marcum Q-function is monotonic, the detection probability is bounded like

$$Q_N\left(\sqrt{2\rho\lambda_{\min}^2 z}, \sqrt{2\eta}\right) \le P_{d|\mathbf{h}} \le Q_N\left(\sqrt{2\rho\lambda_{\max}^2 z}, \sqrt{2\eta}\right). \tag{10}$$

The random variable z is central Chi-squared distributed with $2M$ degrees of freedom and its probability density function is obtained as [5]

$$f_Z(z) = \frac{1}{(M-1)!} z^{M-1} e^{-z}, \; z \ge 0. \tag{11}$$

So the average detection probability of frequency-selective channel is upper bounded by

$$P_d^{\text{up}} = \int_0^{\infty} Q_N\left(\sqrt{2\rho\lambda_{\max}^2 z}, \sqrt{2\eta}\right) f_Z(z)\, dz. \tag{12}$$

The above integral can be solved using [7] with adequate substitutions. So the upper bound of the average detection probability can be obtained as

$$P_d^{\text{up}} = A_1 + \beta^M e^{-\eta} \sum_{i=1}^{M-1} \frac{\eta^i}{i!} \, {}_1F_1\left(M; i+1; \eta\left(1-\beta\right)\right), \tag{13}$$

where $\beta = \frac{1}{1+\rho\lambda_{\max}^2}$, ${}_1F_1(.;.;.)$ is the confluent hypergeometric function [12], $L_i(.)$ is the Laguerre polynomial of degree i [12] and we have

$$A_1 = e^{-\eta\beta}\Bigg\{\beta^{M-1}L_{M-1}\left(-\eta\left(1-\beta\right)\right) + \left(1-\beta\right)\sum_{i=0}^{M-2}\beta^i L_i\left(-\eta(1-\beta)\right)\Bigg\}. \quad (14)$$

Also, the lower bound of the average detection probability can be written as

$$P_d^{\text{low}} = \int_0^\infty Q_N\left(\sqrt{2\rho\lambda_{min}^2 z}, \sqrt{2\eta}\right) f_Z(z)\, dz. \quad (15)$$

P_d^{low} is derived the same as P_d^{up} by substituting $\lambda_{\min}^2$ instead of $\lambda_{\max}^2$ in equation (13). So the average missed detection probability over the frequency-selective channel denoted as $P_{md} = 1 - P_d$, is bounded by

$$1 - P_d^{\text{up}} \leq P_{md} \leq 1 - P_d^{\text{low}}. \quad (16)$$

Now, we derive the diversity order of the average missed detection probability over the frequency-selective channel. According to [6], diversity order of missed detection probability in high SNR regime is defined as the asymptotic slope of the missed detection probability versus SNR plot in dB/dB scale and is quantified by

$$d = -\lim_{\rho\to\infty}\frac{\log P_{md}(\rho)}{\log(\rho)}. \quad (17)$$

For diversity analysis, we approximate P_{md} in high SNR regime. When $\rho \to \infty$, $\beta = \frac{1}{1+\rho\lambda_{\max}^2} \approx \frac{1}{\rho\lambda_{\max}^2}$ and $1 - \beta \approx 1$. So A_1 in equation (14) can be approximated as

$$A_1 \approx e^{-\eta\beta}\sum_{i=0}^{M-1}\beta^i L_i\left(-\eta\right). \quad (18)$$

According to [12], we can write

$$\sum_{i=0}^{M-1}\beta^i L_i\left(-\eta\right) = (1-\beta)^{-1}e^{\frac{-\eta\beta}{\beta-1}} - \sum_{i=M}^{\infty}\beta^i L_i\left(-\eta\right) \approx e^{\eta\beta} - \beta^M L_M\left(-\eta\right). \quad (19)$$

By Substituting (19) in (18), we obtain

$$A_1 \approx 1 - \beta^M L_M\left(-\eta\right) \approx 1 - \left(\frac{1}{\rho\lambda_{\max}^2}\right)^M L_M\left(-\eta\right). \quad (20)$$

So according to (13), the upper bound of the average detection probability can be approximated as

$$P_d^{\text{up}} \approx 1 - \left(\frac{1}{\rho\lambda_{\max}^2}\right)^M \Bigg\{L_M\left(-\eta\right) - e^{-\eta}\sum_{i=1}^{M-1}\frac{\eta^i}{i!}\ {}_1F_1\left(M; i+1; \eta\right)\Bigg\}. \quad (21)$$

Finally, the upper bound of the average detection probability can be expressed as

$$P_d^{\text{up}} = 1 - \frac{B}{\rho^M}, \quad (22)$$

where

$$B = \frac{\left[L_M\left(-\eta\right) - e^{-\eta}\sum_{i=1}^{M-1}\frac{\eta^i}{i!}\,{}_1F_1\left(M; i+1; \eta\right)\right]}{\left(\lambda_{\max}\right)^{2M}}. \quad (23)$$

By the same analysis, we can show $P_d^{\text{low}} = 1 - \frac{C}{\rho^M}$ where C is like B by substituting $\lambda_{\min}^2$ instead of $\lambda_{\max}^2$ in (23). Note that B and C are constant with respect to ρ.

Hence, the average missed detection probability over the frequency-selective channel in high SNR regime is bounded by

$$\frac{B}{\rho^M} \leq P_{md} \leq \frac{C}{\rho^M}. \quad (24)$$

From (24), the diversity order of P_{md} is quantified by

$$d = -\lim_{\rho\to\infty}\frac{P_{md}(\rho)}{\log(\rho)} = M. \quad (25)$$

Thus, the diversity order of the average missed detection probability over the frequency-selective channel is equal to M, i.e., the number of the non-zero singular values of the matrix $\mathbf{X}$. On the other hand, the rank of a matrix is equal to the number of its non-zero singular values. $\mathbf{X}$ is an N by L matrix and its rank is less than or equal to $\min\{L, N\}$. So we can write

$$d = \text{rank}(\mathbf{X}) \leq \min\{L, N\}. \quad (26)$$

It can be seen that the number of the samples (N) and the number of the channel taps (L) affect the rank and thus the diversity order. If $N < L$, then the diversity order is less than L and the multipath diversity for an L-tap channel can not be achieved. If the rank is equal to L which is highly probable when $N >> L$ as a result of the randomness of the primary user signal, the diversity of P_m will be equal to L , i.e., the multipath diversity can be achieved. In this case, the channel with more taps leads to significant decrease in the missed detection probability. So getting enough samples from the primary user signal is necessary to achieve the multipath diversity.

IV. Simulation Results

In this section, the performance of spectrum sensing over frequency-selective channel is evaluated by Monte-Carlo simulation and we confirm our theoretical analysis. In all simulations, we have set $P_f = 0.01$ and found the threshold η. The frequency-selective channel is modelled by L independent random taps. The tap gains are Rayleigh distributed. Then, the average missed detection probability over the channel is simulated for different SNRs.

In all simulations, the primary user signal, $x_p(n)$, is drawn randomly and independently from a binary phase shift keying (BPSK) constellation.

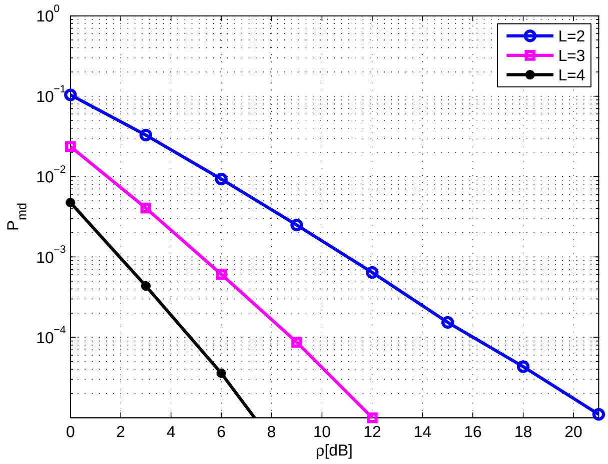

Fig. 1. The average probability of missed detection versus SNR (ρ) for different channel taps (L). In all cases $N = 30$ and $P_f = 0.01$.

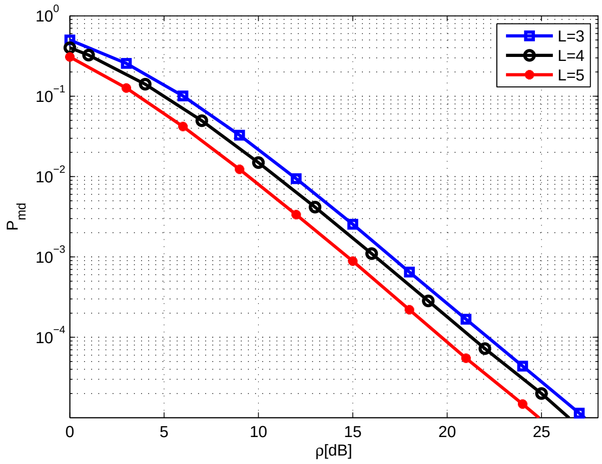

Fig. 3. The average probability of missed detection versus SNR (ρ) for different channel taps (L). In all cases $N = 2$ and $P_f = 0.01$.

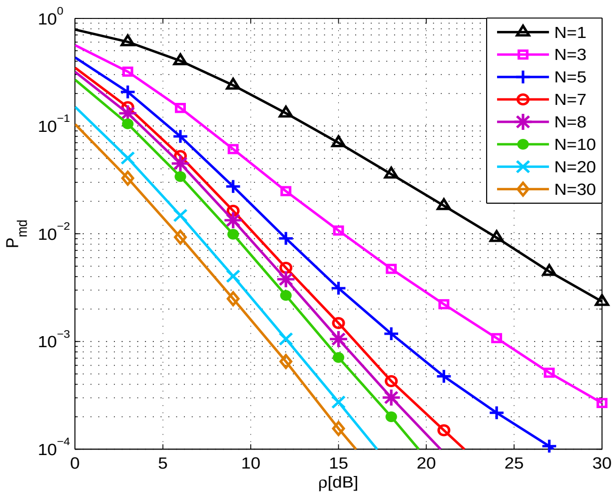

Fig. 2. The average probability of missed detection versus SNR (ρ) for different number of samples (N). In all cases $L = 2$ and $P_f = 0.01$.

Fig. 1 shows the SNR-dependent behaviour of the average missed detection probability for different number of channel taps $L = 2, 3, 4$, when $N = 30$. It can be seen that in this case, the slope of the missed detection curve decreases proportional to $-L$, which means that the diversity order is equal to the number of channel taps. As L increases, the average missed detection probability decreases significantly.

Fig. 2 shows the average probability of missed detection versus SNR (ρ) for different number of samples (N) when $L = 2$. It can be seen that the number of the samples that we get from the primary user's signal affects the missed detection performance. In terms of diversity, for $L = 2$ taps and BPSK signal, to achieve the multipath diversity equal to $L = 2$, we should get at least $N = 8$ samples. On the other hand, increasing the samples more than $N = 8$ does not help the increase in diversity since the slopes of all the curves for $\rho \geq$ 10 dB are the same and equal to -2. Thus, it is better to take less samples to save time for data transmission.

Fig. 3 shows the SNR-dependent behaviour of the average missed detection probability for different number of channel taps $L = 3, 4, 5$, when $N = 2$ and the rank of $\mathbf{X}$ is less than L. To this end, we build the primary user samples in a way that the rank of $\mathbf{X}$ is 2. It can be seen that for $\rho \geq 10$ dB, the slope of all cases is equal to -2, i.e, the diversity order is equal to the rank of $\mathbf{X}$. So increasing the channel taps does not help the increase in diversity. In other words, the multipath diversity of frequency-selective channel can not be achieved in this case.

V. Conclusion

In this paper, we investigated the performance of spectrum sensing using the energy detector over Rayleigh frequency-selective fading channel. We calculated the diversity order of the average missed detection probability and showed that the diversity order is equal to the rank of the primary user's code matrix, i.e., the improvement of spectrum sensing depends upon both the number of the channel taps and the primary user's samples. We showed that by getting enough samples from the primary user's signal, the multipath diversity can be achieved.

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