Exponential Event-based Consensus for General Liner Dynamic Agents

S. Mohammad Noorbakhsh, *IEEE Member* Jafar Ghaisari*, *IEEE Senior Member* Iman Izadi, *IEEE Member*

Deptartment of Electrical and Computer Engineering, Isfahan University of Technology
Isfahan, 84156-83111, Iran
*m_nourbakhsh@ec.iut.ac.ir, *ghaisari@cc.iut.ac.ir, iman.izadi@cc.iut.ac.ir*

***Abstract*—For multi-agent systems (MASs), event-triggered strategies typically require less control task execution and information transmission in achieving a collective behaviour compared to the time-triggered control schemes. This paper investigates exponential event-based consensus in directed topologies for MASs with general linear dynamic agents. The proposed control scheme involves a piecewise continuous event-based control law and an event-based data transmission strategy for each individual agent which are developed based on a co-design procedure. Necessary triggering instants in the agents are determined by local triggering mechanisms (ETMs) based on the violation of well-designed triggering-conditions. To avoid continuous time communication between agents, a time-dependent ETM is proposed. Then, analytical discussions on Zeno behaviour exclusion are given for the proposed ETM. It is proved that the proposed control schemes guarantees exponential consensus achievement meanwhile decreases both control input updates and data broadcasts by the agents. As a result, such control scheme facilitates more energy saving and extends the lifespan of agent actuators in addition to decreasing the communication costs. Illustrative simulations are given to support the effectiveness of the proposed method.**

Keywords— Event-triggered control, Multi-Agent systems, Consensus problem, Strongly connected graph.

I. INTRODUCTION

In the past decade, collective behavior of multi-agent systems have received an increasing attention within the control engineering community due to its broad application in many areas including cooperative control of unmanned air vehicles, autonomous underwater vehicles [1], clock synchronization [2], air traffic control [3], clustering of satellites, sensor networks [4] and so on. As a basic problem, consensus problem of MASs concerns with developing distributed control schemes that enables a group of dynamic agents connected through a communication network to reach an agreement on certain quantities of interest [4, 5].

In practice, each autonomous agent like a mobile robot is often equipped with small digital micro-processor, onboard communication module, and actuation module which usually have limited energy resources, communication resources and computing capabilities to perform the required functions. These factors motivate researchers to investigate event-triggered control schemes [6] which typically requires less control task execution and data broadcasting in achieving a certain level of performance compared to the time-scheduled control schemes. Event-triggered strategies have been recently extended to the consensus of integrator agents in [7-12]. The authors in [7] proposed an event-triggered control law to update local controllers at some triggering instants. However, a continuous monitoring of the states of the neighboring agents was required in [7]. To relax this requirement, [8-11] introduced various ETMs in which continuous monitoring of the neighbour states was no longer needed. More recently, 13-19] have extended the event-triggered consensus for MASs with general linear agents. In [13] and [16-18], the interaction topology of the agents were supposed to be undirected for the aim of easy analysis since the Laplacian matrix is symmetric; while, the information flow may be directed in real-world applications. The consensus problem in a directed interaction topology is more challenging as the stability analysis would be more complicated. The event-triggered consensus problem in directed topologies were investigated in [14, 15] and [19]. The authors in [14] addressed an event-based consensus controller in term of linear matrix inequalities (LMIs); however, the LMIs required explicitly Laplacian matrix L. Moreover, [15] gave sufficient conditions for convergence to a ball centered at the consensus point. However, each agent had to broadcast its control input signals in addition to its sampled-states at the triggering instants which could increase the communication loads. In addition, [14, 15], and [19] proposed state-dependent triggering conditions which may be highly frequent when the number of agents is large. To overcome this limitation we consider the time-dependent event threshold function, for event-based consensus in directed communication networks..

This paper investigates distributed event-triggered consensus control problem of linear MASs in directed topologies. The proposed approach concurrently focuses on decreasing control executions and reducing data broadcasts by the agents while guarantees exponential convergence to consensus. A co-design procedure is provided to determine parameters of distributed consensus controllers and the local time-dependant ETMs, simultaneously. Sufficient conditions ensuring exponential consensus achievement is derived and it is proved that the proposed event-based control scheme will not exhibit Zeno behavior. Following the proposed structure, each agent broadcasts data to neighbors just at its own trigger instants not at the neighbors trigger instants. Such approach makes a great decrease in the required communication bandwidth and also facilitate more energy saving.

An outline of this paper is as follows. Some preliminaries are given in Section II. The proposed event-triggered framework is discussed in Section III. Numerical simulation results are given in Section IV. Finally, Section V presents a summary of conclusions.

II. PRELINIARIES

A. Algebraic graph theory

Algebraic graph theory [20] is often used in the study of collective behaviour of MASs to describe communication between agents.

Consider a directed graph G consisting a set of N nodes $\mathcal{V} = \{v_i\}_{i\in\mathcal{N}=\{1,2,..,N\}}$, a set of directed edges $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$. An edge in the graph denoted by (v_j, v_i) originating at node v_i and ending at node v_j. $\mathcal{A}(G) = [a_{ij}]\epsilon\mathbb{R}^{N\times N}$ represents the adjacency matrix where $a_{ij} = 1$ if $(v_j, v_i)\epsilon\mathcal{E}$ and $a_{ij} = 0$, otherwise. A directed path from node υ_1 to υ_r is a finite ordered sequence of edges, $(\upsilon_{k+1}, \upsilon_k), k = 1,2, \dots, r-1$ such that consecutive nodes are adjacent. The set of in-neighbours of node v_i is denoted by $N_i = \{v_j \in \mathcal{V} : (v_i, v_j)\epsilon\mathcal{E} \ , j \neq i\}$. A directed graph is strongly connected if and only if there exists a directed path between every pair of distinct vertices. A directed graph is called balanced if $\sum_{i=1}^{N} a_{ij} = \sum_{i=1}^{N} a_{ji}$ for all $i \in \mathcal{N}$. The $Laplacian$ of G is defined as $\mathrm{L} = [l_{ij}]\epsilon\mathbb{R}^{N\times N}$, $l_{ij} = -a_{ij}$, $i \neq j$ and $l_{ii} = \sum_{m=1}^{N} a_{im}$ for $i \in \mathcal{N}$. By definition, every row sum of L is zero. Throughout this paper, we assume the directed graph G is strongly connected and balanced.

Lemma [21]. Suppose that directed graph G is strongly connected and balanced. Then, the Laplacian matrix L have both zero row sums and zero column sums. Moreover, zero is an eigenvalue of L and the corresponding eigenvector is the vector of ones, $\underline{1}_N\epsilon\mathbb{R}^{N\times 1}$, i.e. $\underline{1}_N{}^T L = \underline{0}^T$, $L^T\underline{1}_N = \underline{0}$. In addition, $L + L^T$ is a positive semi-definite matrix with zero being its simple eigenvalue. Furthermore, since $L + L^T$ is symmetric and $(L + L^T)\underline{1}_N = \underline{0}$, by *Courant-Fischer minmax theorem* one has that $\min_{x\neq 0_{x\perp\mathbf{1}}} \frac{x^T(L+L^T)x}{x^T x} = \lambda_2\{L + L^T\}$, where $\lambda_2\{L + L^T\} > 0$ is the second smallest eigenvalues of $L + L^T$.

Notations. $\|\cdot\|$ denotes the Euclidean norm for vectors and the induced 2-norm for matrices, respectively. $A\otimes B$ denotes the Kronecker product of matrix A and B.

B. Problem formulation

Consider a network of N agents which communicate over a directed graph G. The dynamic model of agent $i \in \mathcal{N}$ is described by (1) where $x_i(t) \in \mathbb{R}^n$ and $u_i(t) \in \mathbb{R}^m$ are the state and the control input of agent I, respectively.

$$\dot{x}_i(t) = Ax_i(t) + Bu_i(t) \tag{1}$$

The objective is to introduce an event-triggered consensus control scheme such that enables linear MAS (1) to achieve consensus exponentially for any initial state $x(0)\epsilon\mathbb{R}^{nN}$. To solve the consensus problem of MAS (1) in an event-based manner, a local ETM for each agent *i* monitors the state vector of this agent to generate a sequence of triggering time instants $\{t_r^i\}_{r=1,2,...}$ using a prescribed trigging condition. Let t_r^i denotes the triggering instant of r^{th} event in agent *i* .The next trigger instant t_{r+1}^i for agent *i* is determined by the given ETM in (2) whenever the measurement error $e_i(t) = x_i(t_r^i) - x_i(t)$ exceeds a given threshold function $f_i(t) = r_i e^{-\theta_i t}$ with $r_i > 0$ and $0 < \theta_i < 1$.

$$t_{r+1}^i = t_r^i + inf\{\, t\colon \ \| e_i(t) \|^2 > f_i(.)\} \tag{2}$$

Consider consensus error $q_{ij}(t) = x_i(t) - x_j(t)$. Then, vector $q_{ij}(t_r) = x_i(t_r^i) - x_j\left(t^j_{r'(t^-)}\right)$ denotes the value of the consensus error at the triggering instants during the time interval $[t_{r+1}^i, t_r^i)$. $t^j_{r'(t^-)} = max\{\, t^* : \ \ t^* \in \{t^j_{r'}\} \ , \ t^* < t \ \}$ represents the last triggering instant in agent *j* before time t. The event-based control law is as follow

$$u_i(t) = -\alpha K \sum_{j\in N_i} q_{ij}(t_r), \qquad \forall\, t \in [t_r^i, t_{r+1}^i) \tag{3}$$

In (3), matrix $K \in \mathbb{R}^{m\times n}$ and positive scalar α denote the consensus controller gain and coupling strength for agent *i*, respectively. The control input (3) is held constant till the next triggerimg instant in a ZOH manner. Based on (3), the control signal $u_i(t)$ is updated at the trigger instants $\{t_r^i\}_{r=1,2,...}$ in addition to whenever an event is triggered in one of the neighbouring agents i.e. at $\{t^{j\in N_i}_{r'}\}_{r'=1,2,...}$.

The main purpose of this paper is to find the threshold function $f_i(.)$ for $i \in \mathcal{N}$ and consensus controller parameters i.e. controller gain K and coupling strength α in a co-design manner to solve the consensus problem for MAS (1). In addition, it is desirable that each agent broadcasts data to its neighbors just at its own triggering instants to facilitate a reduction in communication costs. Specially, the design methodology should let us to implement local ETM (2) and the event-based control law (3) in a distributed manner for each agent.

III. MAIN RESULTS

A. Exponential Event-based Consensus

From (1) and (3), the dynamic of agent *i* during the time interval $[t_r^i, t_{r+1}^i)$ becomes

$$\dot{x}_i(t) = Ax_i(t) - \alpha BK \sum_{j\in N_i} [x_i(t_r^i) - x_j\left(t^j_{r'(t^-)}\right)] \tag{4}$$

Let $x = [x_1{}^T, \dots, x_N{}^T]^T$ and $e = [e_1{}^T, \dots, e_N{}^T]^T$. Then, the closed loop system model is as follow

$$\dot{x}(t) = (I_N \otimes A - \alpha L\otimes BK)x(t) - (\alpha L\otimes BK)e(t) \tag{5}$$

Assumption 1. The communication topology is a strongly connected and balanced directed graph.

Let $z_i(t) = x_i(t) - \frac{1}{N}\sum_{k=1}^{N} x_k(t)$ denotes the disagreement vector for agent *i*. Then, we have that $z(t) = (\Lambda \otimes I_n)x(t)$ where $\Lambda = I_N - \frac{1}{N}\underline{1}_N\underline{1}_N{}^T$ with $\underline{1}_N = [1\ \ 1\ \ \dots 1]^T$. Thereby, the disagreement vector dynamic is given by (6) where $\varepsilon(t) = (\Lambda \otimes I_n)e(t)$.

$$\dot{z}(t) = (I_N \otimes A - \alpha L \otimes BK)z(t) - (\alpha L \otimes BK)\varepsilon(t) \tag{6}$$

The matrix Λ has a unique zero eigenvalue with $\underline{1}_N$ as its corresponding right eigenvector. Thereby, using $z(t) = (\Lambda \otimes I_n)x(t)$ one has that $x(t) \in span\{\underline{1}_N\}$ if $z(t) = 0_{nN}$. Thereby, the exponential consensus problem of the closed loop MAS (5) is transformed into the exponential stability problem for the disagreement dynamic (6).

Now, we consider the following Lyapunov function candidate where P is a positive definite.

$$V(t) = z^T(I_N \otimes P)z \tag{7}$$

Let $K = B^T P$. Calculating the time derivative of $V(t)$ along the trajectory (6) yields

$$\begin{aligned}\dot{V}(t) &= 2z^T(I_N \otimes P)\dot{z}\\ &= 2z^T(I_N \otimes P)[(I_N \otimes A - \alpha L \otimes BK)z - (\alpha L \otimes BK)\varepsilon]\\ &= 2z^T[I_N \otimes PA - \alpha L \otimes PBK]z - 2z^T(\alpha L \otimes PBK)\varepsilon\\ &= z^T[I_N \otimes (PA + A^T P) - \alpha(L + L^T) \otimes PBB^T P]z\\ &\qquad - 2z^T(L \otimes \alpha PBB^T P)\varepsilon \end{aligned} \tag{8}$$

It can be checked that $z^T(1_N \otimes PB) = 0_{nN}$. Then, by Courant-Fischer minmax theorem and using the facts that $\Lambda^T\Lambda = \Lambda$ and $\Lambda^T L \Lambda = L$, it follows from (8) that

$$\begin{aligned}\dot{V}(t) &= x^T[\Lambda \otimes (PA + A^T P - \alpha\lambda_2 PBB^T P)]x\\ &\qquad - 2x^T(L \otimes \alpha PBB^T P)e\end{aligned} \tag{9}$$

where $\lambda_2 = \min_{i\in\mathcal{N}} \lambda_i\{L + L^T\} > 0$. Note that the eigenvalues set of Λ is $\{0,1,\dots,1\}$. Then, one has that $L^T L \le \tilde{\lambda}\Lambda$ where $\tilde{\lambda}$ is the maximum eigenvalue of $L^T L$. Then, the following inequality in (10) holds for any scalar $r > 0$.

$$\begin{aligned}&x^T(L \otimes \alpha PBB^T P)e\\ &\le \frac{r}{2}x^T(L^T L \otimes I_n)x + \frac{1}{2r}e^T(I_N \otimes \alpha^2(PBB^T P)^2)e\\ &\le \frac{r\tilde{\lambda}}{2}x^T(\Lambda \otimes I_n)x + \frac{\alpha^2\|PBB^T P\|^2}{2r}e^T(I_N \otimes I_n)e\end{aligned} \tag{10}$$

Considering (10), we get from (9) that

$$\begin{aligned}\dot{V}(t) &= x^T\left[\Lambda \otimes (PA + A^T P - \alpha\lambda_2 PBB^T P + \frac{r\tilde{\lambda}}{2}I_n)\right]x\\ &\qquad + \frac{\alpha^2\|PBB^T P\|^2}{2r}\|e\|^2\end{aligned} \tag{11}$$

Assumption 2. Matrix pair (A, B) is stabilizable.

Theorem 3. 1. Consider the MAS (1) with the control law (3) with $K = B^T Q^{-1}$ which is driven by the event-triggering mechanism (2) with $f_i(t) = r_i e^{-\theta_i t}$. Suppose that Assumption 1 and Assumption 2 hold. Then, all agents exponentially achieve consensus if the following conditions in (12), (13) and (14) hold where $\eta < 1$, ρ_1 and ρ_2 are positive scalars.

$$\begin{pmatrix} AQ + QA^T - \rho_1 BB^T + \eta Q & Q \\ Q & -\frac{1}{\rho_2}I_n \end{pmatrix} < 0 \tag{12}$$

$$r \le \frac{2\rho_2}{\tilde{\lambda}} \text{ and } \alpha > \frac{\rho_1}{\lambda_2} \tag{13}$$

$$0 < \underline{\theta} \le \eta \tag{14}$$

Proof. We will prove the exponential stability of (6) which is equivalent to exponential consensus achievement for MAS (1). Based on ETM (2), one has that $\| e_i(t) \|^2 \le r_i e^{-\theta_i(t)}$ during the time interval $t \in [t_r^i, t_{r+1}^i)$. Then, $\| e_i(t) \|^2 \le re^{-\underline{\theta}t}$ where $r = max_{i\in\mathcal{N}}\{r_i\}$. Let $\bar{f}(t) = re^{-\underline{\theta}t}$. Thereby,

$$\| e(t) \|^2 \le N\bar{f}(t), \quad \forall\, t \in [t_r^i, t_{r+1}^i) \tag{15}$$

Let $Q = P^{-1}$. We consider the Lyapunov function candidate (7) which yields (11). Enforcing the derived inequality in (15), we get from (11) that

$$\begin{aligned}\dot{V}(t) &= x^T\left[\Lambda \otimes (PA + A^T P - \alpha\lambda_2 PBB^T P + \frac{r\tilde{\lambda}}{2}I_n)\right]x\\ &\qquad + \frac{\alpha^2\|PBB^T P\|^2}{2r}N\bar{f}(t)\end{aligned} \tag{16}$$

Applying the Schur complement lemma [22] on LMI (12) implies $AQ + QA^T - \rho_1 BB^T + \rho_2 Q^T Q + \eta Q < 0$. Pre- and post-multiplying the derived result by P yields $PA + A^T P - \rho_1 PBB^T P + \rho_2 I_n + \eta P < 0$ which implies $PA + A^T P - \alpha\lambda_2 PBB^T P + \frac{r\tilde{\lambda}}{2}I_n < -\eta P$ if both conditions in (13) hold. Let $\mu = \frac{\alpha^2\|PBB^T P\|^2}{2r}N$. Then, it gets from (16) that

$$\begin{aligned}\dot{V}(t) &\le -\eta x^T(\Lambda \otimes P)x + \mu\bar{f}(t)\\ &= -\eta V(t) + \mu\bar{f}(t)\end{aligned} \tag{17}$$

By the Gronwell inequality, (17) implies

$$\begin{aligned}V(t) &\le V(0)e^{-\int_0^t \eta d\tau} + \int_0^t e^{-\int_\tau^t \eta d\tau}\,\mu\bar{f}(\tau)d\tau\\ &= V(0)e^{-\eta t} + \frac{\mu r}{\eta - \underline{\theta}}[e^{-\underline{\theta}t} - e^{-\eta t}]\end{aligned} \tag{18}$$

From (14), one has that $e^{-\eta t} \le e^{-\underline{\theta}t}$. Then it follows from (18) that

$$V(t) \le [V(0) + \frac{\mu\bar{r}}{\eta - \underline{\theta}}]e^{-\underline{\theta}t} \tag{19}$$

Thus, $\lim_{t\to\infty} V(t) = 0$ and $z(t)$ in the closed loop system (6) exponentially converges to $\underline{0}$. Consequently, all agents in (1) achieve consensus exponentially and the proof is completed.☐

Remark: Assumption 2 is a necessary condition for the solvability of LMI (12) [23].

B. *Zeno-Exclusion*

Zeno behaviour is an important issue which should be excluded in event-triggered control approaches [24]. In order to prove Zeno exclusion by each agent i we need to show that $t_{r+1}^i - t_r^i > 0$.

Theorem 3. 2. Consider the MAS (1) with the control law (6) and suppose that the triggering instants are determined by (2). Then, for any initial state $x(0) \in \mathbb{R}^{nN}$ and any time $t \geq 0$ no agent will exhibit the Zeno behaviour.

Proof: From the results of Theorem 3.1 one could rewrite the Lyapunov function (7) as $V(t) = x^T(\Lambda \otimes P)x \leq [V(0) + \frac{\mu \bar{r}}{\eta - \underline{\theta}}]e^{-\underline{\theta} t}$. Then $\| x(t) \|$ is bounded on $[0, \infty)$ i.e., there exists a positive scalar $\bar{\rho}_x > 0$ such that $\| x(t) \| \leq \bar{\rho}_x$ for $t \geq 0$. Furthermore, $\| e(t) \|$ is also bounded based on the proposed ETM (2). Considering (5), one has that $\| \dot{x} \| \leq \| (I_N \otimes A - \alpha L \otimes BK) \| \| x \| + \| \alpha L \otimes BK \| \| e \|$. As a results, there exists a $\bar{\rho} > 0$ such that $\| \dot{x}(t) \| \leq \bar{\rho}$ for $t \geq 0$. Considering the fact that $\| \dot{x}_i(t) \| \leq \| \dot{x}(t) \|$, one has that $\| \dot{x}_i(t) \| \leq \bar{\rho}$, which yields $\| x_i(t) - x_i(t_r^i) \| \leq \bar{\rho}|t - t_r^i|$. Then, the following upper bound on $\| e_i(t) \|$ holds.

$$\| e_i(t) \| \leq \bar{\rho}|t - t_r^i| \tag{20}$$

The proposed ETM (2) with $f_i(t) = r_i e^{-\theta_i t}$ guarantees that $\| e_i(t) \| \leq \sqrt{f_i(t)}$ for $t \in [t_r^i, t_{r+1}^i)$. One has that

$$\begin{aligned} \sqrt{f_i(t)} &= \sqrt{r_i} e^{-\frac{\theta_i}{2} t} \\ &\geq \sqrt{r_i} e^{-\frac{\theta_i}{2} t} e^{\frac{\theta_i}{2}(t_r^i - \mathcal{T})}, \quad \forall t \in [t_r^i, \mathcal{T}], \mathcal{T} \geq t_{r+1}^i. \\ &= \sqrt{r_i} e^{-\frac{\theta_i}{2}\mathcal{T}} e^{-\frac{\theta_i}{2}(t - t_r^i)} \\ &\geq \sqrt{r_i} e^{-\frac{\theta_i}{2}\mathcal{T}} \left[1 - \frac{\theta_i}{2}(t - t_r^i)\right]. \end{aligned} \tag{21}$$

Let $t = t_{r+1}^i$, it yields form (21) that

$$\sqrt{f_i(t_{r+1}^i)} \geq \sqrt{r_i} e^{-\frac{\theta_i}{2}\mathcal{T}} [1 - \frac{\theta_i}{2}(t_{r+1}^i - t_r^i)] \tag{22}$$

Now, suppose that agent i triggers at t_r^i. The next triggering instants, t_{r+1}^i, would be triggered by ETM (2) whenever the value of $\| e_i(t) \|$ reaches the threshold $\sqrt{f_i(t)}$ at $t = t_{r+1}^i$. Thus, it is concluded from (20) and (22) that

$$\bar{\rho}|t_{r+1}^i - t_r^i| \geq \sqrt{r_i} e^{-\frac{\theta_i}{2}\mathcal{T}} [1 - \frac{\theta_i}{2}(t_{r+1}^i - t_r^i)] \tag{23}$$

Consequently

$$t_{r+1}^i - t_r^i \geq \frac{\sqrt{r_i} e^{-\frac{\theta_i}{2}\mathcal{T}}}{\bar{\rho} + \frac{\theta_i}{2}\sqrt{r_i} e^{-\frac{\theta_i}{2}\mathcal{T}}} \equiv \tau_{r,i}^* \tag{24}$$

It is proved that each agent $i \in \mathcal{N}$ excludes the Zeno behaviour as we always have $t_{r+1}^i - t_r^i \geq \tau_{r,i}^* > 0$. Therefore, the proof is completed. □

Table 1. The proposed co-design procedure

Co-design Procedure: Given MAS in (1) with stabilizable dynamic agents which communicate over a strongly connected and balanced graph, the proposed event-based consensus controller in terms of (2) and (3) can be constructed in the following steps.

Step 1- Take a positive scalars ρ_1, ρ_2 *and* $\eta < 1$.

Step 2- Solve LMI (12) to get one feasible matrix $Q = Q^T > 0$.

Step 3-Compute $K = B^T Q^{-1}$ *and choose* $\alpha > \frac{\rho_1}{\lambda_2}$.

Step 4- For each agent i, choose $0 < r_i \leq \frac{2\eta}{\bar{\lambda}}$ *and* $0 < \theta_i < \eta$.

C. *Co-Design Procedure*

In most existing results, a consensus controller were developed first based on an assumption that the agents interact over a perfect communication network, and then an event-based transmission scheme were designed to guarantee the consensus achievement under the event-based strategy. Though, one important challenge in the event-based schemes is to co-design the local ETMs and the distributed control laws so as to enable agents achieving consensus with as few resource utilization as possible.

Using the results from Theorem 3.1 and 3.2, we give the following co-design procedure in Table 1 to solve the event-based consensus problem of linear MAS (1).

IV. Simulation Results

In this section, a simulation example is given to illustrate the effectiveness of the analytical results. Consider four agents whose dynamics are described by

$$\dot{x}_i(t) = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -0.3 & 0.4 \\ 0 & -0.4 & -0.3 \end{bmatrix} x_i(t) + \begin{bmatrix} 0.1 \\ 0 \\ -0.1 \end{bmatrix} u_i(t) \tag{25}$$

One can check that the matrix pair (A, B) is controllable The communication topology is shown in Fig 1 which is a strongly connected and balanced directed graph. We choose $\eta = 1$, $\rho_1 = 0.1$ and $\rho_2 = 1$. Solving LMI (12) by using the SeDuMi toolbox [25] gives a solution

$$Q = \begin{bmatrix} 0.3918 & 0.0166 & -0.0254 \\ 0.0166 & 0.0491 & -0.0025 \\ -0.0254 & -0.0025 & 0.0510 \end{bmatrix}.$$

Using Theorem 3.1, the feedback gain matrix in the control law (3) is derived as $K = [0.1381 \quad -0.1436 \quad -1.8990]$. We set the coupling gain in the control law (3) as $\alpha = 0.2$ and the local ETM parameters as $r_1 = 0.09$, $r_2 = 0.12$, $r_3 = 0.11$, $r_4 = 0.1$ and $\theta_i = 0.2$ for $i = 1,2,3,4$ which satisfy the required conditions (13) and (14). The initial conditions for numerical simulation are randomly chosen from interval $[-1, 1]$. Furthermore, it is supposed that the system achieve consensus whenever $max_{i,j} \| x_i(t) - x_j(t) \| \leq 10^{-4}$. Numerous simulations are conducted and the main results are discussed in the following. Fig. 2 illustrates the state trajectories of the agents under the proposed event-triggered consensus controller, from which it can be observed that the agents achieve consensus. Note that each agent communicates with neighbors and updates its control input just at the triggering instants. In Fig. 2, the markers present the broadcasted samples to the agents. Furthermore, the variations of the measurement error norm $\parallel e_i(t) \parallel^2$ during the numerical simulation are given in Fig. 3 for each agent i. The figure shows that $\parallel e_i(t) \parallel^2$ is always upper bounded by threshold function $f_i(t) = r_i e^{-\theta_i t}$ and it resets to zero on event instants $\{t_r^i\}_{r=1,2,\ldots}$. A brief report of the simulation results is given in Table 2 where for each agent i, $\tau_{min\,i} = min\{t_{r+1}^i - t_r^i\}_{r=1,2,\ldots}$ and $\tau_{max_i} = max\{t_{r+1}^i - t_r^i\}_{r=1,2,\ldots}$ are defined as the minimum and maximum inter-event time interval, respectively. The average inter-event time interval is denoted by $\tau_{mean\,i}$. The numbers of triggers generated by the local ETMs in agent 1, 2, 3, 4 are 12, 11, 10, and 9, respectively. The event triggering instants are given in a separate plot in Fig. 4. Furthermore, Table 2 presents the number of samples that each agent sent to its neighbours and also the number of its control signal updates before achieving consensus. For example, the second agent broadcaste 24 samples, 12 samples to third agent and 12 samples to forth agent. Moreover, the number of updates in control input $u_2(t)$ is 33. Because, this control input would be updated at $\{t_r^2\}_{r=1,2,..,11} \cup \{t_r^1\}_{r=1,2,..,12} \cup \{t_r^3\}_{r=1,2,..,10}$.

In the following, we give a comparison between our proposed approach in Theorem 3.1 and the ones in [19] which has addressed event-based consensus in strongly connected and balanced graphs. The required design parameters in each approach are chosen in order to generate nearly the same settling time. Table 3 gives the results derived by [19]. Comparing the presented results in Table 2 and Table 3 demonstrates the superiority of the proposed control scheme in Theorem 3.1 in term of reduction in number of broadcasts by each agent and also in term of control executions in each agent. By applying the proposed event-based approach in Theorem 3.1, all agent broadcasts 75 samples through the communication network to their neighbours while the agent broadcasts 275 samples when the given consensus protocols by [19] is employed. Then, Theorem 3.1 makes a significant reduction in the communication network utilization and also facilitates more energy saving than [19]. Furthermore, from Table 3, the agents achieve consensus with 183 control signal updates while, it is 432 control updates when the given method in [19] is applied. Thus, the agents could achieve consensus with less control executions and less data broadcasting which facilitate more energy saving and decreasing actuators wears by applying the proposed co-design approach in Table 1.

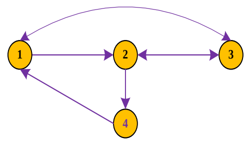

Fig. 1. Communication network topology.

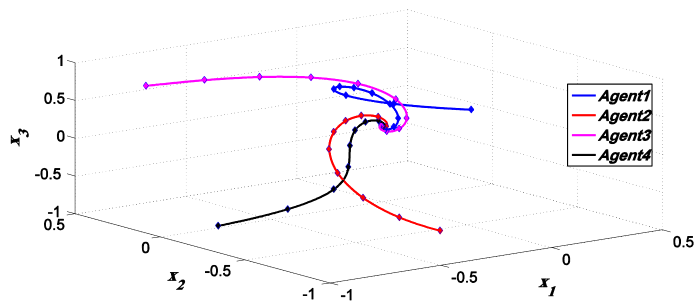

Fig. 2. State trajectories of agents. The markers represent the broadcasted samples to the neighbours .

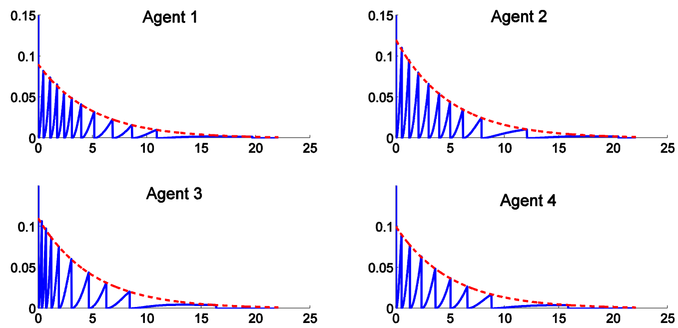

Fig. 3. Evolutions of $\parallel e_i(t) \parallel^2$. The threshold function $f_i(t) = r_i e^{-\theta_i t}$ is shown by dotted line for each agent i .

Table 2. Simulation results via the proposed event-triggered consensus controller in terms of (2) and (3)

	Agent 1	*Agent 2*	*Agent 3*	*Agent 4*
$\tau_{min\,i}$, *(sec.)*	0.46	0.53	0.31	0.50
τ_{max_i}, *(sec.)*	8.89	8.42	7.96	7.05
τ_{mean_i}, *(sec.)*	1.67	1.90	1.66	1.81
Total number of event triggers	12	11	10	9
Number of broadcasted samples	24	22	20	9
Number of control input update	31	33	33	20

Table 3. Simulation results by applying the proposed event-triggered consensus controller in [19].

	Agent 1	*Agent 2*	*Agent 3*	*Agent 4*
τ_{mean_i}, *(sec.)*	0.37	0.40	0.47	0.44
Total number of event triggers	39	48	31	39
Number of broadcasted samples	78	96	62	39
Number of control input update	109	118	118	87

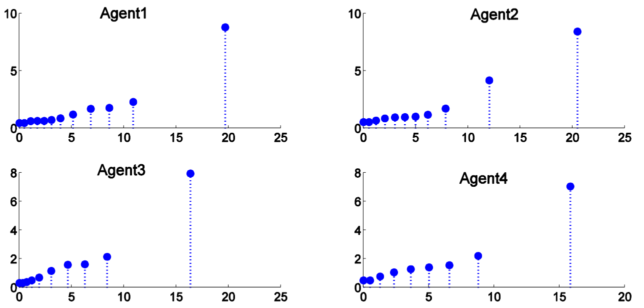

Fig. 4. In each plot, the horizental axe indicates the triggering instants and the vertcal axe denotes the length of inter-event time intervals , $\{t_{r+1}^i - t_r^i\}$,in seconds, for the realted agent.

V. Conclusion

The paper has investigated a distributed event-triggered approach to address the consensus problem of general linear agents in strongly connected and balanced topologies. A time-dependant event-triggering strategy has been first proposed for each agent and then sufficient conditions have been provided to achieve consensus exponentially. It has been proved that the proposed local ETMs did not introduce Zeno behaviour in the closed loop systems as the inter-event time intervals were lower-bounded by a positive constant for each individual agent. A co-design procedure has been given to determine the required design parameters for the investigated event-based control law and local ETMs. Following the proposed control scheme, each individual agent required to broadcast data and to execute its control input just at the triggering instants. Consequently, it led to a significant reduction in the communication network utilization and energy consumption in addition to extend the lifespan of the actuators. Simulation results illustrated the effectiveness of the proposed method. Future work will involve extending the proposed event-driven approach to address the H_∞ consensus problem for nonlinear dynamic agents in directed topologies.

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