A Low Complexity Hardware for Compressive Sensing Matrix Generation

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Abstract— **In this paper a low complexity hardware is designed to generate a deterministic matrix for compressive sensing systems. The construction of the matrix is based on the connection between the parity check matrix of LDPC codes and the measurement matrix of compressive sensing. For efficient hardware realization, a geometric approach to the construction of LDPC codes is used for matrix generation on the fly without requiring a lot of storage. The performance of generated matrix is approved under ℓ_1-minimization and OMP recovery algorithms, and it performs comparably to the corresponding random matrix. The described hardware has been implemented on a Xilinx Spartan 6 FPGA.**

Keywords- Hardware Design, Compressive Sensing, Finite Geometry, Sparse Approximation

I. Introduction

Many natural and man-made signals have a sparse representation on an appropriate basis [1]. Sampling these signals at nyquist rate produces a large redundancy in sampled data. Thus it is necessary to compress data before its storage or transmission. In compressive sensing framework, signals are sampled in the compression form [2]–[5]. This approach is widely used in the practical applications such as magnetic resonance imaging (MRI) [6], radar [7], wireless communication [8], [9], and monitoring the electrical activities of the human body [10]–[12]. The important part of this method of data acquisition is the measurement matrix that exhibits a dimensional reduction [2]. An appropriate measurement matrix should satisfy some criteria to guaranty that the original signal can be recovered accurately[13]. In most hardware implementation frameworks, especially for high speed signal reconstruction, a random measurement matrix is employed [9], [11], [14]–[18], but using these matrices have some limitations including their storage requirements and computational cost [13]. These drawbacks make random matrices inefficient for hardware implementation. Deterministic construction of measurement matrix is another approach that alleviates the mentioned problems [13], [19]–[24]. Since the entries of deterministic matrices can be computed on the fly, these matrices provide storage efficiency [19]. Also, by using the structure of deterministic matrices, the recovery algorithms can be accomplished with lower complexities and lower memory access.

In recent years, various deterministic measurement matrices are presented in the literatures, e.g., Toeplitz and circulant matrices [25], sparse binary matrices [10], [13], Chirp sensing codes [22], finite fields [21] and second order Reed-Muller codes [20]. On the other hand, hardware implementations of several recovery algorithms are presented in the literatures [11], [14]–[18]. However, most of these works use random measurement matrices that are hard to be implemented in hardware. This limits the practical usage of reconstruction algorithms especially when the size of signal is large.

In this paper a novel hardware architecture for realizing low-complexity deterministic construction of measurement matrix is designed and implemented on a FPGA platform. The proposed method is based on the constructing codes via finite geometry [26], [27]. An important property of the finite geometry codes is that they are either cyclic or quasi cyclic [26]. This feature leads to a rotational shift architecture that can be implemented in linear time. The structure of proposed deterministic construction helps to have fast recovery in real time applications.

This paper organized as follows. The preliminaries of compressive sensing are introduced in section II. Section III describes the matrix construction details. Hardware Implementation and Simulation results are presented in sections IV and V respectively. And finally, the conclusions are provided in section VI.

II. Preliminaries

From mathematical point of view, sampling in compressive sensing can be represented by multiplying the k-sparse signal $x \in \mathbb{R}^n$ with a measurement matrix $A \in \mathbb{R}^{m\times n}$ [2]. Signal x is called k-sparse if it has only k nonzero entries with $k \ll n$. Matrix A maps an input signal with large number of samples into a typically much smaller signal denoted by y, i.e., the measured vector $y \in \mathbb{R}^m$ is obtained as below:

$$y=Ax \tag{1}$$

An under-determined set of linear equations should be solved for reconstructing x from the above measurement. This problem seems to be ill-posed. But, by selecting a suitable measurement matrix under certain conditions and If the signal x is known to be sparse, then a unique solution exists. It is proven that a sparse signal can be exactly recovered by solving a sparse approximation problem. The problem can be expressed as follows [2], [28]:

$$\min \|x\|_0 \quad \text{subject to } Ax=y, \tag{2}$$

where $\|x\|_0 \triangleq |\{i: x_i \neq 0\}|$ denotes the ℓ_0-quasi norm of x. Solving this problem requires an exhaustive enumeration of all possible sparse patterns and hence it is NP-hard. There are two conventional methods in compressive sensing for relaxing this problem that greatly reduce the required computational effort. One pursues a convex optimization or the ℓ_1-minimzation problem that can be stated in a relaxed form as a basis pursuit as follows [4]:

$$\min \|x\|_1 \quad \text{subject to } Ax=y, \tag{3}$$

where $\|x\|_1 \triangleq \Sigma\, |x_i|$ (i=1...n) denotes the ℓ_1-norm of x. The other approach considers greedy algorithms that typically produce sub-optimal solutions. The required computation in the greedy algorithms is smaller than that of convex relaxations. These algorithms, such as the matching pursuit (MP) and its modifications, operate in an iterative search manner and the optimal solution is approximated well.

To solve the sparse problem in above mentioned methods, the measurement matrix should satisfy some criteria. The first criterion is the Null Space Property (NSP). Null space of a matrix A is defined as [2]:

$$\mathcal{N}(A)=\{z : Az = 0\} \tag{4}$$

Matrix A uniquely represents all k-sparse x if and only if $\mathcal{N}(A)$ contains no vectors in the 2k-sparse space. One of the most common methods for characterizing this property is based on a parameter known as the spark. Spark (A) is the minimal number of linearly dependent columns of A defined as [29]:

$$\text{spark }(A)=\min \{\|z\|_0 : z \in \mathcal{N}(A))\} \tag{5}$$

It has been shown that if and only if spark (A) >2k, for any vector y there exists at most one k-sparse signal such that y=Ax [2].

The other important property in matrix construction is known as Restricted Isometry Property (RIP). For any k-sparse signal x, a matrix $A_{M\times N}$ fulfills the RIP of order k with restricted isometry constant δ_k if [13]:

$$(1-\delta_k)\|x\|_2^2 \le \|Ax\|_2^2 \le (1+\delta_k)\|x\|_2^2 \tag{6}$$

By satisfying this condition, any k columns vectors from A behave like an almost orthogonal system. While the RIP guarantees exact and robust recovery of sparse signals, there is not any efficient way to verify that a general matrix A satisfies this property [23]. In many cases it is preferable to use instead a property of A that is easily computable [2]. The coherence of a matrix is one such property, that its small value implies RIP. The coherence of a matrix A is defined as [13]:

$$\mu(A) = \max_{i,j\in[1,n],i\neq j} \frac{|\langle a_i, a_j \rangle|}{\|a_i\|_2 \|a_j\|_2} \tag{7}$$

where $\langle \cdot,\cdot \rangle$ is the inner product of any two columns of A, and $\|a\|_2 \triangleq (\Sigma\ (x_i)^2)^{1/2}$ (i=1..n) denotes the ℓ_2-norm of an n-dimensional vector. The coherence parameter measures the smallest angle between each pair of matrix columns and plays an important role in the deterministic matrix construction. It is shown that the coherence of a matrix is always in the range of [2]:

$$\sqrt{\frac{n-m}{m(n-1)}} \le \mu(A) \le 1 \tag{8}$$

The lower bound is known as the Welch bound, and when $n \gg m$, it approximately equals $1/\sqrt{m}$. There are some relations between the introduced properties, that permits to easily compute or at least estimate them by using the parameter used to design the measurement matrix. For example, the coherence and spark properties of a matrix is related by [29]:

$$\text{spark}(A) \ge 1 + \frac{1}{\mu(A)} \tag{9}$$

III. Matrix Construction

In [27], a mathematical connection between LDPC codes and compressive sensing is illustrated, and linear programming decoding in channel coding and compressive sensing are linked together. The linear programming is extensively used for the problem of finding the sparsest solution of an underdetermined system of linear equations. An LDPC code is defined as the null space of a parity check matrix denoted by H [19], [26], [27], [29]. In the following, an algebraic method is presented for constructing a modified parity check matrix based on hyperplanes of two different dimensions in finite geometries.

For a prime number p and two integers m and s ($m\geq 2$ and $s\geq 1$), the m-dimensional Euclidean geometry over finite field $GF(p^s)$ is represented by $EG(m,p^s)$, and m-tuples over $GF(p^s)$ are the points of this geometry [26]. If α is a primitive element of a finite field $GF(p^{ms})$, then the elements in $GF(p^{ms})$ can be represented as powers of α as follows: $\alpha^{-\infty}=0, \alpha^0=1, \alpha^1, \alpha^2, \ldots, \alpha^{p^{ms}-2}$, where 0 represents the origin of the geometry. Every element α^i in $GF(p^{ms})$ can be expressed as [30]:

$$\alpha^i = a_{i0} + a_{i1}\alpha + a_{i2}\alpha^2 + \cdots + a_{i,m-1}\alpha^{m-1} \tag{10}$$

where $a_{ij} \in GF(p^s)$ for $0\leq j<m$. It follows from (10) that there is a one-to-one correspondence between the element α^i and the m-tuple $(a_{i0}, a_{i1}, \ldots, a_{i,m-1})$ over $GF(p^{ms})$. Thus the elements of $GF(p^{ms})$ forms an m-dimensional Euclidean geometry $EG(m,p^s)$. A μ-dimensional subspace of the vector space of all the m-tuples over $GF(p^{ms})$ is called a μ-flat. A μ-flat can be represented as [30]:

$$a_0 + \beta_1 a_1 + \beta_2 a_2 + \cdots + \beta_\mu a_\mu \tag{11}$$

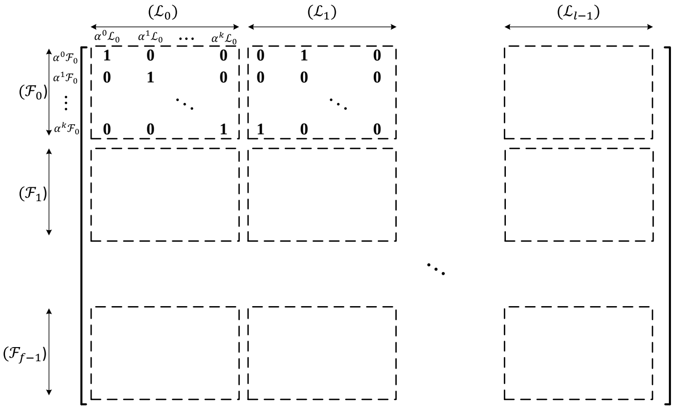

Figure 2. Partioning incidence matrix into disjiont cyclic classes

where β_i is an element in the finite field GF(p^s). By beta variations, the points belong to a certain μ-flat are obtained and each set of point form a μ-flat that pass through the α_0 point. By selection of different points in (11), all the μ-flats are determined over GF(p^{ms}). A set of μ-flats which are disjoint and contain all the points of EG(m,p^s), are parallel to each other and form a parallel μ-flat bundle. A parallel bundle contains all the points of EG(m,p^s) where each point appearing once and only once [29].

For $0 \le \mu_1 < \mu_2 \le m$ the parity check or incidence matrix of μ_2-flat over μ_1-flat is a binary matrix that it's rows corresponds to all the μ_2-flats and it's columns corresponds to all the μ_1-flats in EG(m,p^s). The entry h_{ij} of this matrix is '1' if and only if the ith μ_2-flat contains the jth μ_1-flat. Every row of this matrix is an incidence vector of μ_2-flats over μ_1-flats [26], [29]. Assume EG*(m,p^s) denotes the EG(m,p^s) without origin and all the flats passing through the origin. It has been shown that all the μ-flats in EG*(m,p^s) can be partitioned into disjoint cyclic classes [26]. The μ-flats of each class can be arranged in such a way that each incidence vector corresponding to a certain μ-flat is a cyclic shift of the incidence vector above it. This grouping of incidence vectors in the rows and columns of the constructed matrix leads to the square submatrices that each of which is a circulant matrix. This structure is shown in Fig. 1.

To obtain a circulant matrix in which each row vector is rotated one element to the right relative to the preceding row vector, a parity check matrix is constructed based on the lines and points of EG*(m,p^s). By arranging the columns of this matrix in order of $\alpha^0=1$ α^1, α^2,… , $\alpha^{p^{ms}-2}$ where α is a primitive

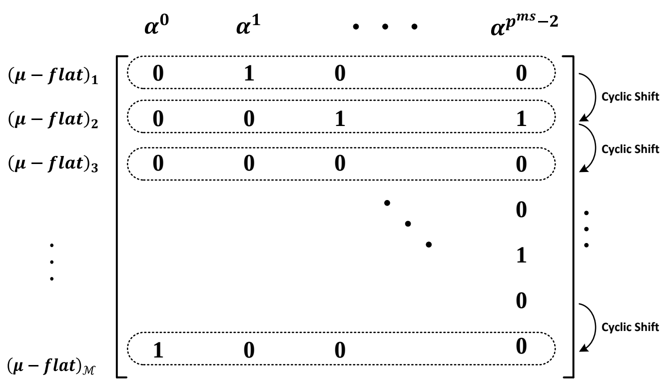

Figure 4. Circulant incidence matrix

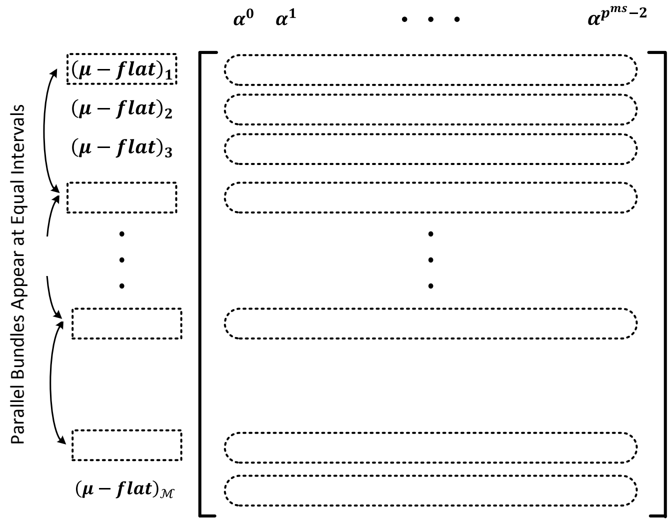

Figure 1. Parallel bundles in the circulant incidence matrix structure

element of GF(p^{ms}) and also arranging the rows of this matrix in term of the cyclic classes a circulant matrix is generated as shown in Fig. 2. By selecting a suitable number of the parallel bundles, the circulant matrix is appropriated for dimensionality reduction in compressive sensing systems [29]. After arranging the matrix, parallel bundles appear at equal intervals and can be removed systematically. Fig. 3 shows the parallel bundles in the circulant incidence matrix structure.

according to (7) and (8), the coherence and the Welch bound of the proposed matrix is 0.125 and 0.055, respectively. In [29] an improved lower bound of spark is obtained for binary matrix from finite geometry that shows their relatively large spark. the large value of spark gives an intuitively description on the good empirical performance of the constructed matrix by this method.

IV. Hardware Implementation

The proposed architecture for the hardware implementation of the deterministic matrix generation is shown in Fig. 4. First a 1023×1023 parity check matrix is constructed based on the not passing through the origin lines and non-origin points of

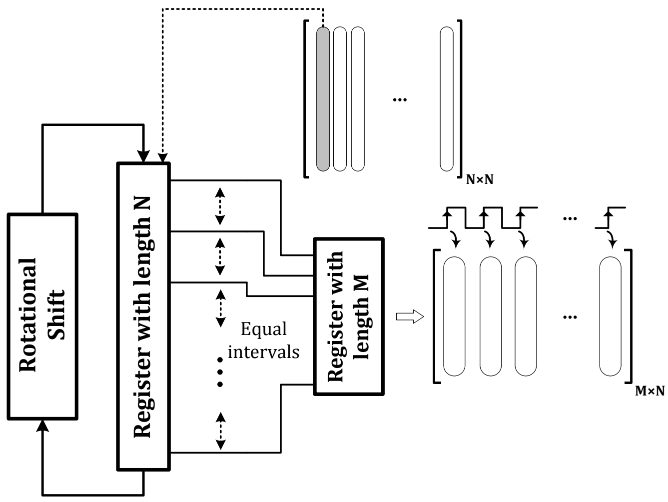

Figure 3. Deterministic matrix generation hardware

EG($2,2^5$). Then by selection of 8 parallel bundles, the matrix is made appropriate for dimensionality reduction in compressive sensing systems.

As shown in Fig. 4, one column of square matrix is put in a register, and in every clock, the data is shifted rotationally to the down. Through a proper selection of the elements, the matrix is generated. The element selection is based on the parallel structure concept of Euclidian geometry. As mentioned before, after reordering the points in EG based on the primitive elements of the finite field, some arrangement appears in parallel bundles at equal intervals. In other words, μ-flats that belong to a specific parallel bundle, repeat at a certain intervals. By selection of the corresponding entry from shifted row, the columns of matrix are constructed in each clock. The proposed design is implemented on a Spartan6 FPGA. The device utilization summary and the timing obtained from the place and route are presented in Table I.

V. Simulation Results

For performance evaluation, a simulation is carried out under the following conditions. A 248×1023 measurement matrix is constructed based on the finite geometry, and to compare with random measurement matrices, a same size Gaussian matrix is also used. The entries of the random matrix are chosen independent and identically distributed (i.i.d.) from pseudorandom numbers. A k-sparse signal is obtained by first selecting the places of the nonzero elements by using random permutation of the integers, and then generating the elements values from normally distributed pseudorandom numbers. The size of the signal is 1023 which is sampled by the constructed matrix to a 248 measured signal. Fig. 5 shows the sparsity effect on the recovery algorithm for both Gaussian random matrix and the proposed deterministic matrix. The average percentage of perfect OMP and ℓ_1 minimization recovery over 1000 trials is shown in this diagram. The relative recovery error is calculated as $e=\|x^*-x\|_2/\|x\|_2$, where x^* represents the recovered signal. If $e<0.001$, the recovery is considered to be perfect. As depicted in Fig. 5, the performance of the recovery algorithm using deterministic matrix is comparable to the random matrix.

VI. Conclusion

In this paper a low complexity hardware for deterministic generation of compressive sensing measurement matrix is proposed. The matrix construction is based on the line-point incidence matrix of the Euclidian Geometry. This matrix has a sparse, binary, and cyclic structure that provides an efficient hardware realization without requiring a considerable storage space. The designed hardware can generate every column of the matrix on the fly within a period of about 1ns. The binary and sparse properties of the presented matrix can help to speed up the recovery algorithms in real time applications. The simulation results show that the accuracy of the recovered signal by using the proposed measurement matrix is comparable to the random matrix.

TABLE I. Implementation Results for Xilinx Spartan6 FPGA

Design Specifications	
Max Frequency (MHz)	800
Occupied Slices	8 (1%)
Slice LUTs	32 (1%)
Dynamic Power Consumption (mW)	0.5

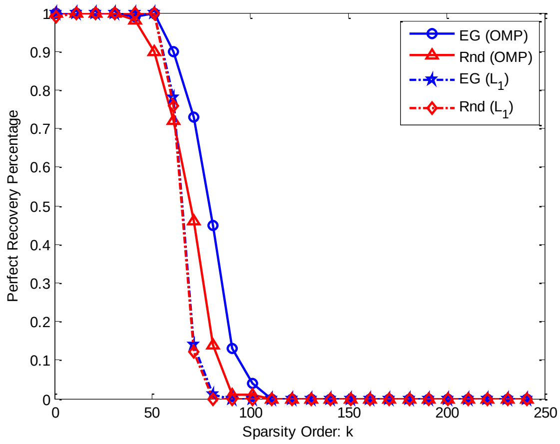

Figure 5. Sparsity effect on the recovery algorithm for both Gaussian random matrix and the proposed deterministic matrix.

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