**Singular Perturbation theory in Control of Nonlinear Systems with Matched and Unmatched Uncertainties**

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Abstract— **In this paper, a state-feedback controller is proposed for stabilization of a class of nonlinear systems in the presence of matched and unmatched uncertainties. By combination of backstepping and time scale separation, first, to deal with the existence of uncertainties, high-gain filters are designed, which estimate the uncertainties and then, a fast dynamical equation is derived where the solution is sought to approximate the corresponding ideal virtual/actual control inputs. In this approach the problem of "explosion of complexity” caused by the traditional backstepping technique is eliminated. Finally, the simulation results are provided to demonstrate the effectiveness of the proposed approach.**

***Keywords-component; singular perturbation theory; unmatched uncertainty; backstepping; explosion of complexity.***

# Introduction

The control of dynamical systems, whose mathematical models contain uncertainties, has occupied the attention of researchers in recent times and has been extensively studied [1-12]. Chakrabortty and Arcak [1] proposed time-scale separation based robust redesign technique for stabilization of uncertain nonlinear systems. In [1], a high gain filter is designed to estimate the uncertainty. Control design is based on time-scale separation using tools of the theory of singular perturbations [5, 9-10]. The fast variable arising from this filter is used in the nominal feedback control law to cancel the effect of the uncertainy. So, after a fast transient the closed loop trajectories converge to the nominal trajectories.

In [1], the control approach and Lyapunov redesign discussed for nonlinear systems with uncertainty satisfying the matching condition, that is, when it appears in the same equation as the control. The matching condition assumption is unfortunately fairly restrictive and not satisfied by the majority of real world systems. Hence, the non-matching disturbances or uncertainty, that is, when it appears before the control input may cause unacceptable deterioration in the performance of the regulated output. In [13], the work of Chakrabortty and Arcak [1] is extended to system with unmatched uncertainty by employing time scale separation in backstepping procedure. Backstepping method is one of the most popular techniques of nonlinear control design [5, 7-8,11] which provides a systematic framework. However, because of the repeated differentiations of the virtual control inputs in the backstepping procedure, the problem of "explosion of complexity” has been occurred. In [14, 15], this problem is solved by time scale separation.

In this paper, by employing the time scale separation method in the backstepping procedure first, high gain filters are designed to estimate the uncertainties. Then, the time derivatives of the virtual/actual control inputs are defined as solutions of fast dynamic equations and their integrals are used as the virtual/actual control inputs. This approach can overcome the uncertainties and also the problem of "explosion of complexity” caused by the traditional backstepping technique is eliminated

This paper is organized as follows. Preliminary results as well as problem formulation are presented in section 2. In section 3, we develop the controller structure. Finally the simulation results and some conclusion remarks are given in sections 4 and 5.

# PRELIMINARIES AND PROBLEM FORMULATION

## Preliminiries on singular perturbation theory [5]

Consider the problem of solving the state equation

(1)

where and are smooth. It is assumed that the functions and are continuously differentiable in their arguments for where and are open connected sets, . If has isolated real roots *,* for each when , we say that the model (1) is in ‘standard form’. Let us choose one fixed parameter }, and drop the subscript *a* from *h* from now on. Let where denotes a chosen root of *l* roots satisfying . From singular perturbation theory, the ‘reduced system’ is represented by

(2)

and the ‘boundary layer system’ with the new time scale is defined as

(3)

where and are treated as fixed parameters. The following Tikhonov ̉s theorem is introduced [5]

**Theorem 1**: Consider the singular perturbation system (1), and let be an isolated root of . Assume that the following conditions are satisfied for all for some domains and , which contain their respective origins.

(A1) On any compact subset of , the functions *f* and *g*, their first partial derivatives with respect to and the first partial derivative of *g* with respect to *t* are continuous and bounded. and have bounded first partial derivatives with respect to their arguments, and is Lipschitz in *x*, uniformly in *t*, and the initial data and are smooth functions of .

(A2) The origin is an exponentially stable equilibrium point of the reduced system (2). There exists a Lyapunov function that satisfies

(4)

(5)

for all , where are continuous positive-definite functions on , and let *c* be a non-negative number such that is a compact subset of .

(A3) The origin is an exponentially stable equilibrium point of the boundary layer system (3), uniformly in *.* Let be the region of attraction of and let be a compact subset of . Then, for each compact set , there exists a positive constant such that for all *t ≥ 0, ,* , and , (1) has a unique solution on and holds uniformly forwhere is the solution of the reduced system (2).

**Remark 1:** Assumption (A3) in Theorem 1 can be verified locally via linearization [5]. It can be shown that if there exists such that the Jacobian matrix satisfies the eigenvalue condition for all , then Assumption (A3) is satisfied.

## Problem statement

Consider the following uncertain system

(6)

where , and are the system states, control input and system output respectively. and are uncertain nonlinearities. It is noted that are unmatched uncertainties and is matched uncertainty. , , are continuously differentiable non-linear functions in their arguments.

The objective of this paper is to design a tracking control law for the nonlinear system (6) such that the output follows the desired trajectory .

**Assumption 1**: and are either positive or negative. Without lossing the generality, we assume and .

# main results

## Controller design

The control is developed by combination of backstepping, and singular perturbation theory. Similar to the backstepping method, this design procedure contains *n* steps. Employing time-scale separation concept, the unknown uncertainties and virtual control laws , *i = 1, . . . , n − 1* are obtained at each step. Finally, the actual control law *u* is designed at step *n*. The design procedure is presented in the following. Introduce the change of coordinates and where

*Step 1*. We start with the first equation of (6)

(7)

First, to estimate the unknown , we design the filter

(8)

where *.* Then from (7) and (8), the variable

(9)

Satisfies

(10)

When is small, evolves in a faster time scale than , and reaches a small neighborhood of the manifold

(11)

by considering as the control variable. The derivative of is given as

(12)

Then, as the first virtual controller can be specified as the solution of

(13)

resulting in the asymptotically stable closed-loop dynamics for the first subsystem. > 0 is the first control gain. According to the following fast dynamics based on time-scale separation concept, an approximate virtual controller is designed

(14)

with the initial condition *, ,*

(15)

Where from (11), is replaced by .

Let be an isolated root of Then the reduced system is defined as

(16)

and the boundary layer system can be represented by

(17)

(18)

Where , , and .

Considering the control Lyapunov function and using the reduced system (16), it is deduced that

(19)

*Step* : The derivative of is expressed as

(20)

Similar to step 1, first to estimate the unknown , we design the filter

(21)

where *.* Then from (20) and (21), the variable

(22)

Satisfies

(23)

When is small, evolves in a faster time scale than , and reaches a small neighborhood of the manifold

(24)

we should find such that

(25)

where is the th positive control gain. In this step, the time derivative of the virtual control input is appeared which has been designed in the previous step . Therefore, the “explosion of complexity” arising from the calculation of this term is avoided.

The th approximate virtual controller can be designed as the followingth fast dynamic

(26)

with the initial condition *, ,*

(27)

Where from (24), is replaced by .

Let be an isolated root of Then, the reduced system is defined as

(28)

and the boundary layer system can be represented by

(29)

(30)

Where , , and .

Considering the control Lyapunov function and using the reduced system (28), it is deduced that

(31)

*Step n***:** In the last step, the actual control input appears and is at our disposal. We derive the dynamics

(32)

we design the filter

(33)

where *.* From (32) and (33), the variable

(34)

Satisfies

(35)

When is small, evolves in a faster time scale than , and reaches a small neighborhood of the manifold

(36)

we now obtain an approximate actual control input via time-scale separation to satisfy

(37)

as

(38)

with the initial condition  *,*and

(39)

*.*  is the *n*th positive control gain.

Let be an isolated root of Then the reduced system is defined as

(40)

and the boundary layer system can be represented by

(41)

(42)

Where ,, and .

Considering the control Lyapunov function and using the reduced system (40), it is deduced that

(43)

## Stability analysis

For the stability analysis of the proposed control system, we present the following theorem using Tikhonov’s theorem.

**Theorem2:** Consider the singular perturbation problem of the system (6) and the controllers (14), (26), (38). Assume that the following conditions are satisﬁed for all for some domains and , which contain their respective origins, whereand .

() On any compact subset of , the functions , their first partial derivatives with respect to and the first partial derivative of with respect to are continuous and bounded. Also and have bounded first derivatives with respect to their arguments, ) is Lipschitz in , uniform in *t*.

(B2) and are bounded below by some positive constant for all .

Then, the origins of (17), (18), (29), (30), (41) and (42) are exponentially stable. Besides, let be a compact subset of ,where , is the region of attraction of the autonomous system . Moreover, let be a compact subset of ,where , is the region of attraction of the autonomous system with . Then, for each compact subset , there exist positive constant , and such that for all , , , and , the system of (6), (14), (26) and (38) has the unique solution on , and holds uniformly for

**Proof:** For the use of Tikhonov’s theorem, it should be verified that the conditions in our theorem satisfy assumptions (A1), (A2) and (A3). First, Assumption (B1) directly implies that Assumption (A1) holds. Second, we can show easily that Assumption (A2) holds because the origins of the reduced system (16), (28) and (40) are exponentially stable equilibrium points, that is, for and for some where . From the converse Lyapunov theorem, it follows that there exists a Lyapunov function such that

(43)

(44)

where , , are positive constants and denotes a diagonal matrix. We note that any positive can be chosen in Assumption (A2), and so can be any compact subset of .

Finally, we show from Remark 1 that assumption (A3) holds. The exponential stability of the boundary layer system (17), (18), (29), (30), (41) and (42) can be easily obtained locally by linearization with respect to and . Using Assumption 1 and (B2) yields

(45)

(46)

This implies that the boundary layer system has a locally exponentially stable origin. Therefore, we can apply Tikhonov’s theorem. Accordingly, for each compact subset , there exist positive constant , and such that for all , , , and , the system of (6), (14), (26) and (38) has the unique solution on , and holds uniformly for

# simulation results

To validate the effectiveness of the proposed control approach, consider the following nonlinear system in the presence of both matched and unmatched uncertainties.

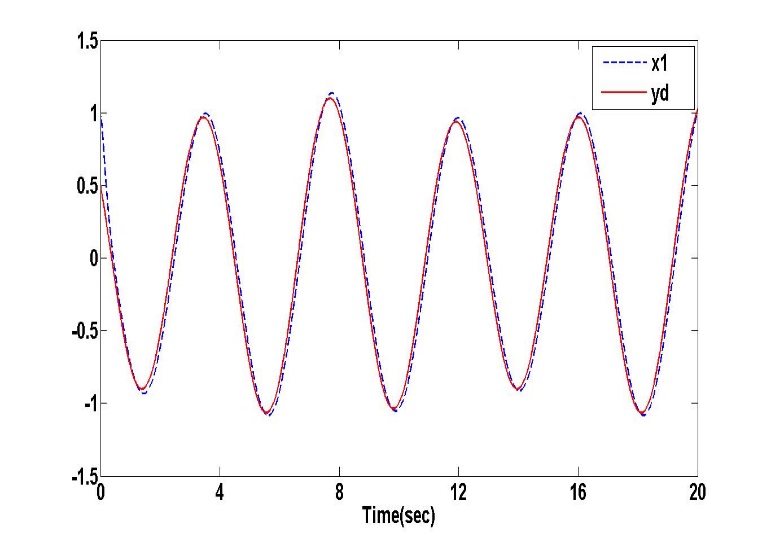
(47)

Where is unmatched and is matched uncertainty of the system.

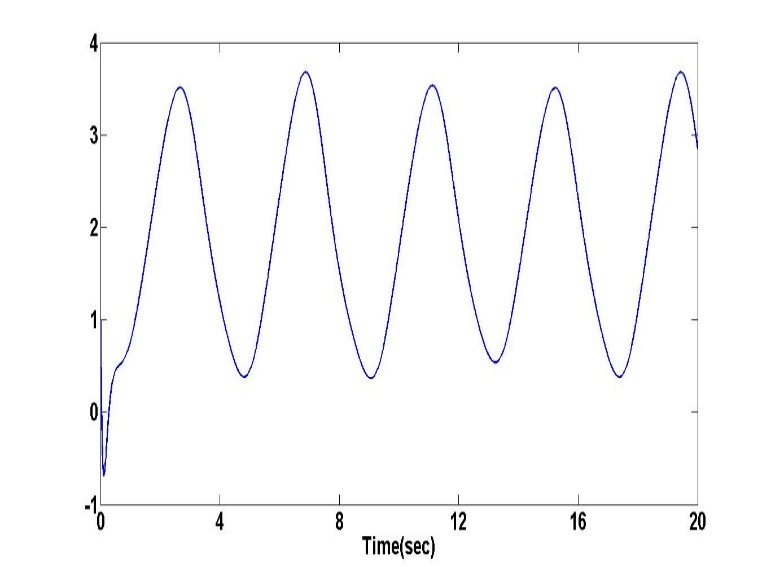
The control object is to synthesize an adaptive control law *u* for system (47) such that the state tracks the reference signal .

The initial conditions set to , , , , , and the design parameters for the proposed control system are adopted as follows: .

Figs. 1-3 show the tracking performance, state trajectory of and the control input, respectively. These figures reveal that the proposed approach has the good control and tracking performance regardless matched and unmatched uncertainties. In addition, note that the states and the control input in the controlled closed-loop system are bounded.



##### Figure 1. Tracking performance



##### Figure 2. state trajectory of

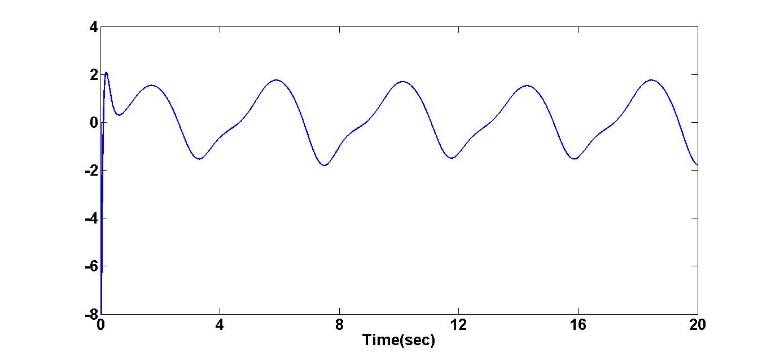


Figure 3. Control input

# conclusion

In this paper, time-scale separation has been employed in backstepping procedure for stabilization of a class of nonlinear systems in the presence of matched and unmatched uncertainties. This approach can overcome the uncertainties and also eliminates the problem of “explosion of complexity” caused by the traditional backstepping technique. Based on Tikhonov’s theorem in singular perturbation theory, the closed loop stability has been proved. The proposed controller guarantees the boundedness of all the signals in the closed-loop system, while the output of the system tracks the desired signal with bounded error.

REFRENCES

[1] A. Chakrabortty, M. Arcak, “Time-scale separation redesigns for stabilization and performance recovery of uncertain nonlinear systems,” Automatica, vol.45, pp. 34-44, 2009.

[2] E. D. Sontag, “Smooth stabilization implies coprime factorization,” IEEE Transactions on Automatic Control, vol. 34, pp. 435-443, 1989.

[3] R. A. Freeman, P. V. Kokotovic, “Robust nonlinear control design,” Boston, MA: Birkhauser, 1996.

[4] R. Marino, P. Tomei, “Nonlinear control design: Geometric, adaptive and robust,” UK: Prentice Hall, 1995.

[5] H. K. Khalil, “Nonlinear systems,” (3rd ed.), Upper Saddle River, NJ: Prentice Hall, 2002.

[6] P. V. Kokotovic, M. Arcak, “Constructive nonlinear control: A historical perspective,” Automatica, vol. 37, no. 5, pp. 637-662, 2001.

[7] Z. Jiang, A. Teel, L. Praly, “Small-gain theorem for ISS systems and applications,” Mathematics of Control, Signals, and Systems, vol. 7, pp. 95–120, 1994.

[8] M. Krstic, I. Kanellakopoulos, P. V. Kokotovic, “Nonlinear and adaptive control design,” New York: Wiley, 1995.

[9] L. B. Freidovich, H. K. Khalil, “Robust feedback linearization using extended high-gain observers,” in Proceedings of conference on decision and control, 2006.

[10] S. Seshagiri, H. K. Khalil, “Robust output feedback regulation of minimum- phase nonlinear systems using conditional integrators,” Automatica, vol. 41, pp. 43-54, 2005.

[11] C.C. Cheng, G.L. Su, C.W. Chien, “Block backstepping controllers design for a class of perturbed non-linear systems with m blocks,” IET Control Theory Appl., vol. 6, pp. 2021–2030, 2012.

[12] S. S. Ge, C. C. Hang, T. H. Lee, T. Zhang, “Stable Adaptive Neural Network Control,” The Netherlands: Kluwer Academic Publishers, pp. 37-38, 2002.

[13] S. P, A. KH, “Backstepping Time-scale Separation Redesign for Stabilization of a Class of Nonlinear Systems with Matched and Unmatched Uncertainties,” Electrical Engineering (ICEE), , pp. 1-6, 2013.

[14] S. J. Yoo, “Adaptive control of non-linearly parameterized pure-feedback systems”, IET Control Theory Appl, vol. 6, iss. 3, pp. 467–473, 2012.

[15] D. Gao, Z. Sun, and B. Xu, “Fuzzy adaptive control for pure-feedback system via time scale separation”, International Journal of Control, Automation, and Systems, vol. 11, no. 1, pp. 147-158, 2013.