# Singular Perturbation theory in Control of Nonlinear Systems with Matched and Unmatched Uncertainties

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*Abstract*— **In this paper, a state-feedback controller is proposed for stabilization of a class of nonlinear systems in the presence of matched and unmatched uncertainties. By combination of backstepping and time scale separation, first, to deal with the existence of uncertainties, high-gain filters are designed, which estimate the uncertainties and then, a fast dynamical equation is derived where the solution is sought to approximate the corresponding ideal virtual/actual control inputs. In this approach the problem of "explosion of complexity" caused by the traditional backstepping technique is eliminated. Finally, the simulation results are provided to demonstrate the effectiveness of the proposed approach.**

***Keywords-component; singular perturbation theory; unmatched uncertainty; backstepping; explosion of complexity.***

## I. INTRODUCTION

The control of dynamical systems, whose mathematical models contain uncertainties, has occupied the attention of researchers in recent times and has been extensively studied [1-12]. Chakrabortty and Arcak [1] proposed time-scale separation based robust redesign technique for stabilization of uncertain nonlinear systems. In [1], a high gain filter is designed to estimate the uncertainty. Control design is based on time-scale separation using tools of the theory of singular perturbations [5, 9-10]. The fast variable arising from this filter is used in the nominal feedback control law to cancel the effect of the uncertainy. So, after a fast transient the closed loop trajectories converge to the nominal trajectories.

In [1], the control approach and Lyapunov redesign discussed for nonlinear systems with uncertainty satisfying the matching condition, that is, when it appears in the same equation as the control. The matching condition assumption is unfortunately fairly restrictive and not satisfied by the majority of real world systems. Hence, the non-matching disturbances or uncertainty, that is, when it appears before the control input may cause unacceptable deterioration in the performance of the regulated output. In [13], the work of Chakrabortty and Arcak [1] is extended to system with unmatched uncertainty by employing time scale separation in backstepping procedure. Backstepping method is one of the most popular techniques of nonlinear control design [5, 7-8,11] which provides a systematic framework. However, because of the repeated differentiations of the virtual control inputs in the backstepping procedure, the problem of "explosion of complexity" has been occurred. In [14, 15], this problem is solved by time scale separation.

In this paper, by employing the time scale separation method in the backstepping procedure first, high gain filters are designed to estimate the uncertainties. Then, the time derivatives of the virtual/actual control inputs are defined as solutions of fast dynamic equations and their integrals are used as the virtual/actual control inputs. This approach can overcome the uncertainties and also the problem of "explosion of complexity" caused by the traditional backstepping technique is eliminated

This paper is organized as follows. Preliminary results as well as problem formulation are presented in section 2. In section 3, we develop the controller structure. Finally the simulation results and some conclusion remarks are given in sections 4 and 5.

## II. PRELIMINARIES AND PROBLEM FORMULATION

### A. *Preliminiries on singular perturbation theory [5]*

Consider the problem of solving the state equation

$$\dot{x}(t) = f(t, x(t), z(t), \varepsilon) \qquad x(0) = \xi(\varepsilon)$$

$$\varepsilon\dot{z}(t) = g(t, x(t), z(t), \varepsilon) \qquad z(0) = \eta(\varepsilon) \qquad (1)$$

where $\xi(\varepsilon)$ and $\eta(\varepsilon)$ are smooth. It is assumed that the functions $f$ and $g$ are continuously differentiable in their arguments for $(t, x, z, \varepsilon) \in [0, \infty) \times D_x \times D_z \times [0, \varepsilon_0]$ where $D_x \subset R^n$ and $D_z \subset R^m$ are open connected sets, $\varepsilon_0 > \gg 0$ . If $g(t, x, z, 0) = 0$ has $l \geq 1$ isolated real roots $z = h_a(t, x)$, $a = 1, 2, \dots, l$ , for each $(t, x) \in [0, \infty) \times D_x$ when $\varepsilon = 0$ , we say that the model (1) is in 'standard form'. Let us choose one fixed parameter $a \in \{1, 2, \dots, l\}$, and drop the subscript $a$ from $h$ from now on. Let $v = z - h(t, x)$ where $h(t, x)$ denotes a chosen root of $l$ roots satisfying $g(t, x, z, 0) = 0$. From singular perturbation theory, the 'reduced system' is represented by

$$\dot{x}(t) = f(t, x(t), h(t, x(t)), 0), \qquad x(0) = \xi(0) \tag{2}$$

and the 'boundary layer system' with the new time scale $\tau = t/\varepsilon$ is defined as

$$\frac{dv}{d\tau} = g\big(t, x, v + h\big(t, x(t)\big), 0\big), \qquad v(0) = \eta_0 - h(0, \xi_0) \tag{3}$$

where $\eta_0 = \eta(0)$ and $\xi_0 = \xi(0)$ are treated as fixed parameters. The following Tikhonov's theorem is introduced [5]

**Theorem 1**: Consider the singular perturbation system (1), and let $z = h(t, x)$ be an isolated root of $g(t, x, z, 0) = 0$. Assume that the following conditions are satisfied for all $(t, x, z - h(t, x)\ , \varepsilon) \in [0, \infty) \times D_x \times D_v \times [0, \varepsilon_0]$ for some domains $D_x \subset R^n$ and $D_v \subset R^m$, which contain their respective origins.

(A1) On any compact subset of $D_x \times D_v$, the functions $f$ and $g$, their first partial derivatives with respect to $(x, z, \varepsilon)$ and the first partial derivative of $g$ with respect to $t$ are continuous and bounded. $h(t, x)$ and $[\partial g(t, x, z, 0)/\partial z]$ have bounded first partial derivatives with respect to their arguments, and $[\partial f\ (t, x, h(t, x), 0)/\partial x]$ is Lipschitz in $x$, uniformly in $t$, and the initial data $\xi(\varepsilon)$ and $\eta(\varepsilon)$ are smooth functions of $\varepsilon$.

(A2) The origin is an exponentially stable equilibrium point of the reduced system (2). There exists a Lyapunov function $V(t, x)$ that satisfies

$$W_1(x) \le V(t, x) \le W_2(x) \tag{4}$$

$$\frac{\partial V}{\partial t}(t, x) + \frac{\partial V}{\partial x}(t, x) f(t, x, h(t, x), 0) \le -W_3(x) \tag{5}$$

for all $(t, x) \in [0, \infty) \times D_x$, where $W1, W2, W3$ are continuous positive-definite functions on $D_x$, and let $c$ be a non-negative number such that $\{x \in D_x | W_1(x) \le c\}$ is a compact subset of $D_x$.

(A3) The origin is an exponentially stable equilibrium point of the boundary layer system (3), uniformly in $(t, x)$. Let $R_v \subset D_v$ be the region of attraction of $(\partial v/\partial \tau) = g(0, \xi_0, v + h(0, \xi_0), 0)$, and let $\Omega_v$ be a compact subset of $R_v$. Then, for each compact set $\Omega_x \subset \{x \in D_x | W_2(x) \le \rho c, 0 < \rho < 1\}$, there exists a positive constant $\varepsilon^*$ such that for all $t \ge 0$, $\xi_0 \in \Omega_x$, $\eta_0 - h(0, \xi_0) \in \Omega_v$, and $0 < \varepsilon < \varepsilon^*$, (1) has a unique solution $x(t, \varepsilon)$ on $[0, \infty)$, and $x(t, \varepsilon) - x_r(t) = O(\varepsilon)$ holds uniformly for $t \in [0, \infty)$, where $x_r(t)$ is the solution of the reduced system (2).

**Remark 1:** Assumption (A3) in Theorem 1 can be verified locally via linearization [5]. It can be shown that if there exists $\varphi_0 > 0$ such that the Jacobian matrix $(\partial g/\partial y)$ satisfies the eigenvalue condition $Re[\lambda\{\partial g(t, x, v + h(t, x), 0)/\partial v\}] \le -\varphi_0 < 0$ for all $(t, x) \in [0, \infty) \times D_x$, then Assumption (A3) is satisfied.

## *B. Problem statement*

Consider the following uncertain system

$$\dot{x}_i = f_i(\bar{x}_i) + g_i(\bar{x}_i)(x_{i+1} + \delta_i(\bar{x}_i)) \qquad i = 1, \dots, n-1$$

$$\dot{x}_n = f_n(\bar{x}_n) + g_n(\bar{x}_n)(u + \delta_n(\bar{x}_n))$$

$$y = x_1 \tag{6}$$

where $\bar{x}_i = [x_1, x_2, \dots, x_i]^T \in R^i$, $u \in R$ and $y \in R$ are the system states, control input and system output respectively. $\delta_i$ and $\delta_n$ are uncertain nonlinearities. It is noted that $\delta_i$ are unmatched uncertainties and $\delta_n$ is matched uncertainty. $f_i: D_{\bar{x}_i} \to R$, $g_i: D_{\bar{x}_i} \to R$, $f_n: D_{\bar{x}_n} \to R$, $g_n: D_{\bar{x}_n} \to R$ are continuously differentiable non-linear functions in their arguments.

The objective of this paper is to design a tracking control law $u$ for the nonlinear system (6) such that the output $y$ follows the desired trajectory $y_d$.

**Assumption 1**: $g_i(\bar{x}_i)$ and $g_n(\bar{x}_n)$ are either positive or negative. Without lossing the generality, we assume $g_i(\bar{x}_i) > 0$ and $g_n(\bar{x}_n) > 0$.

# III. MAIN RESULTS

## *A. Controller design*

The control is developed by combination of backstepping, and singular perturbation theory. Similar to the backstepping method, this design procedure contains *n* steps. Employing time-scale separation concept, the unknown uncertainties and virtual control laws $\alpha_i$, $i = 1, \dots, n-1$ are obtained at each step. Finally, the actual control law $u$ is designed at step $n$. The design procedure is presented in the following. Introduce the change of coordinates $z_1 = x_1 - y_d$ and $z_i = x_i - \alpha_{i-1}$ where $i = 2, \dots, n$.

*Step 1.* We start with the first equation of (6)

$$\dot{z}_1(t) = f_1(x_1) + g_1(x_1)\big(x_2 + \delta_1(x_1)\big) - \dot{y}_d \tag{7}$$

First, to estimate the unknown $\delta_1(x_1)$, we design the filter

$$\dot{\hat{z}}_1(t) = f_1(x_1) + g_1(x_1)(x_2 - \frac{1}{\varepsilon_{1,1}}(\hat{z}_1 - z_1)) - \dot{y}_d$$

$$\hat{z}_1(0) = z_1(0) \tag{8}$$

where $\varepsilon_{11} \ll 1$. Then from (7) and (8), the variable

$$l_1 = \frac{\hat{z}_1 - z_1}{\varepsilon_{1,1}} \tag{9}$$

Satisfies

$$\varepsilon_{1,1}\dot{l}_1 = g_1(x_1)(-l_1 - \delta_1(x_1)) \tag{10}$$

When $\varepsilon_{1,1}$ is small, $l_1$ evolves in a faster time scale than $z_1$, and reaches a small neighborhood of the manifold

$$l_1 = -\delta_1(x_1) \tag{11}$$

by considering $x_2$ as the control variable. The derivative of $z_1$ is given as

$$\dot{z}_1(t) = f_1(x_1) + g_1(x_1)(z_2 + \alpha_1 + \delta_1(x_1)) - \dot{y}_d \tag{12}$$

Then, $\alpha_1$ as the first virtual controller can be specified as the solution of

$$f_1(x_1) + g_1(x_1)(z_2 + \alpha_1 + \delta_1(x_1)) - \dot{y}_d = -k_1 z_1 \tag{13}$$

resulting in the asymptotically stable closed-loop dynamics $\dot{z}_1 = -k_1 z_1$ for the first subsystem. $k_1 > 0$ is the first control gain. According to the following fast dynamics based on time-scale separation concept, an approximate virtual controller is designed

$$\varepsilon_{1,2}\dot{\alpha}_1 = -sign\left(\frac{\partial Q_1}{\partial \alpha_1}\right) Q_1(t, \bar{z}_2, \alpha_1, l_1) \tag{14}$$

with the initial condition $\alpha_1(0) = \alpha_{1,0}$, $\varepsilon_{1,2} \ll 1$, $\bar{z}_2 = [z_1, z_2]^T$

$$Q_1(t, \bar{z}_2, \alpha_1, l_1) = k_1 z_1 + f_1(x_1) + g_1(x_1)(z_2 + \alpha_1 - l_1) - \dot{y}_d \tag{15}$$

Where from (11), $\delta_1(x_1)$ is replaced by $-l_1$.

Let $\alpha_1 = h_1(t, \bar{z}_2, l_1)$ be an isolated root of $Q_1(t, \bar{z}_2, \alpha_1, l_1) = 0$. Then the reduced system is defined as

$$\dot{z}_1 = -k_1 z_1 \qquad z_1(0) = z_{1,0} \tag{16}$$

and the boundary layer system can be represented by

$$\frac{dy_{11}}{d\tau_{11}} = g_1(x_1)(-y_{11}) \tag{17}$$

$$\frac{dy_{12}}{d\tau_{12}} = -sign\left(\frac{\partial Q_1}{\partial \alpha_1}\right) Q_1(t, \bar{z}_2, y_{12} + h_1(t, \bar{z}_2, l_1), l_1) \tag{18}$$

Where $y_{11} = l_1 + \delta_1(x_1)$, $y_{12} = \alpha_1 - h_1(t, \bar{z}_2, l_1)$, $\tau_{11} = \frac{t}{\varepsilon_{1,1}}$ and $\tau_{12} = \frac{t}{\varepsilon_{1,2}}$.

Considering the control Lyapunov function $V_1 = \frac{1}{2} z_1^2$ and using the reduced system (16), it is deduced that

$$\dot{V}_1 = -k_1 z_1^2 \tag{19}$$

*Step i* ($i = 2, \ldots, n-1$): The derivative of $z_i$ is expressed as

$$\dot{z}_i(t) = f_i(\bar{x}_i) + g_i(\bar{x}_i)(x_{i+1} + \delta_i(\bar{x}_i)) - \dot{\alpha}_{i-1} \tag{20}$$

Similar to step 1, first to estimate the unknown $\delta_i(\bar{x}_i)$, we design the filter

$$\dot{\hat{z}}_i(t) = f_i(\bar{x}_i) + g_i(\bar{x}_i)(x_{i+1} - \frac{1}{\varepsilon_{i,1}}(\hat{z}_i - z_i)) - \dot{\alpha}_{i-1}$$

$$\hat{z}_i(0) = z_i(0) \tag{21}$$

where $\varepsilon_{i1} \ll 1$. Then from (20) and (21), the variable

$$l_i = \frac{\hat{z}_i - z_i}{\varepsilon_{i,1}} \tag{22}$$

Satisfies

$$\varepsilon_{i,1}\dot{l}_i = g_i(\bar{x}_i)(-l_i - \delta_i(\bar{x}_i)) \tag{23}$$

When $\varepsilon_{i1}$ is small, $l_i$ evolves in a faster time scale than $z_i$, and reaches a small neighborhood of the manifold

$$l_i = -\delta_i(\bar{x}_i) \tag{24}$$

we should find $\alpha_i$ such that

$$f_i(\bar{x}_i) + g_i(\bar{x}_i)\big(x_{i+1} + \delta_i(\bar{x}_i)\big) - \dot{\alpha}_{i-1} = -k_i z_i \tag{25}$$

where $k_i > 0$ is the $i$th positive control gain. In this step, the time derivative of the virtual control input $\dot{\alpha}_{i-1}$ is appeared which has been designed in the previous step $\dot{\alpha}_{i-1} = -sign\left(\frac{\partial Q_{i-1}}{\partial \alpha_{i-1}}\right) Q_{i-1}(t, \bar{z}_i, \alpha_{i-1}, l_i)/\varepsilon_{i-1,2}$ . Therefore, the "explosion of complexity" arising from the calculation of this term is avoided.

The $i$th approximate virtual controller can be designed as the following $i$th fast dynamic

$$\varepsilon_{i,2}\dot{\alpha}_i = -sign\left(\frac{\partial Q_i}{\partial \alpha_i}\right) Q_i(t, \bar{z}_{i+1}, \alpha_i, l_i) \tag{26}$$

with the initial condition $\alpha_i(0) = \alpha_{i,0}$ , $\varepsilon_{i,2} \ll 1$ , $\bar{z}_{i+1} = [z_1, \ldots, z_{i+1}]^T$

$$Q_i(t, \bar{z}_{i+1}, \alpha_i, l_i) = k_i z_i + f_i(\bar{x}_i) + g_i(\bar{x}_i)((z_{i+1} + \alpha_i - l_i) - \dot{\alpha}_{i-1} \tag{27}$$

Where from (24), $\delta_i(\bar{x}_i)$ is replaced by $-l_i$.

Let $\alpha_i = h_i(t, \bar{z}_{i+1}, l_i)$ be an isolated root of $Q_i(t, \bar{z}_{i+1}, \alpha_i, l_i) = 0$. Then, the reduced system is defined as

$$\dot{z}_i = -k_i z_i \qquad z_i(0) = z_{i,0} \tag{28}$$

and the boundary layer system can be represented by

$$\frac{dy_{i1}}{d\tau_{i1}} = g_i(\bar{x}_i)(-y_{i1}) \tag{29}$$

$$\frac{dy_{i2}}{d\tau_{i2}} = -sign\left(\frac{\partial Q_i}{\partial \alpha_i}\right) Q_i(t, \bar{z}_{i+1}, y_{i2} + h_i(t, \bar{z}_{i+1}, l_i), l_i) \tag{30}$$

Where $y_{i1} = l_i + \delta_i(\bar{x}_i)$, $y_{i2} = \alpha_i - h_i(t, \bar{z}_{i+1}, l_i)$, $\tau_{i1} = \frac{t}{\varepsilon_{i,1}}$ and $\tau_{i2} = \frac{t}{\varepsilon_{i,2}}$.

Considering the control Lyapunov function $V_i = V_{i-1} + \frac{1}{2} z_i^2$ and using the reduced system (28), it is deduced that

$$\dot{V}_i = -\sum_{j=1}^{i} -k_j z_j^2 \tag{31}$$

*Step n*: In the last step, the actual control input $u$ appears and is at our disposal. We derive the $z_n$ dynamics

$$\dot{z}_n = f_n(\bar{x}_n) + g_n(\bar{x}_n)(u + \delta_n(\bar{x}_n)) - \dot{\alpha}_{n-1} \tag{32}$$

we design the filter

$$\dot{\hat{z}}_n(t) = f_n(\bar{x}_n) + g_n(\bar{x}_n)(u - \frac{1}{\varepsilon_{n,1}}(\hat{z}_n - z_n)) - \dot{\alpha}_{n-1}$$

$$\hat{z}_n(0) = z_n(0) \tag{33}$$

where $\varepsilon_{n1} \ll 1$. From (32) and (33), the variable

$$l_n = \frac{\hat{z}_n - z_n}{\varepsilon_{n,1}} \tag{34}$$

Satisfies

$$\varepsilon_{n,1} \dot{l}_n = g_n(\bar{x}_n)(-l_n - \delta_n(\bar{x}_n)) \tag{35}$$

When $\varepsilon_{n1}$ is small, $l_n$ evolves in a faster time scale than $z_n$, and reaches a small neighborhood of the manifold

$$l_n = -\delta_n(\bar{x}_n) \tag{36}$$

we now obtain an approximate actual control input via time-scale separation to satisfy

$$f_n(\bar{x}_n) + g_n(\bar{x}_n)\left(u + \delta_n(\bar{x}_n)\right) - \dot{\alpha}_{n-1} = -k_n z_n \tag{37}$$

as

$$\varepsilon_{n,2} \dot{u} = -sign\left(\frac{\partial Q_n}{\partial u}\right) Q_n(t, \bar{z}_n, u, l_n) \tag{38}$$

with the initial condition $u(0) = u_0$, $\varepsilon_n \ll 1$ and

$$Q_n(t, \bar{z}_n, u, l_n) = f_n(\bar{x}_n) + g_n(\bar{x}_n)(u - l_n) - \dot{\alpha}_{n-1} \tag{39}$$

$\bar{z}_n = [z_1, \ldots, z_n]^T$. $k_n$ is the *n*th positive control gain.

Let $u = h_n(t, \bar{z}_n, l_n)$ be an isolated root of $Q_n(t, \bar{z}_n, u, l_n) = 0$. Then the reduced system is defined as

$$\dot{z}_n = -k_n z_n \qquad z_n(0) = z_{n,0} \tag{40}$$

and the boundary layer system can be represented by

$$\frac{dy_{n1}}{d\tau_{n1}} = g_n(\bar{x}_n)(-y_{n1}) \tag{41}$$

$$\frac{dy_{n2}}{d\tau_{n2}} = -sign\left(\frac{\partial Q_n}{\partial u}\right) Q_n(t, \bar{z}_n, y_{n2} + h_n(t, \bar{z}_n, l_n), l_n) \tag{42}$$

Where $y_{n1} = l_n + \delta_n(\bar{x}_n)$, $y_{n2} = u - h_n(t, \bar{z}_n, l_n)$, $\tau_{n1} = \frac{t}{\varepsilon_{n,1}}$ and $\tau_{n2} = \frac{t}{\varepsilon_{n,2}}$.

Considering the control Lyapunov function $V_n = V_{n-1} + \frac{1}{2} z_n^2$ and using the reduced system (40), it is deduced that

$$\dot{V}_n = -\sum_{j=1}^{n} k_j z_j^2 \tag{43}$$

*B. Stability analysis*

For the stability analysis of the proposed control system, we present the following theorem using Tikhonov's theorem.

**Theorem2:** Consider the singular perturbation problem of the system (6) and the controllers (14), (26), (38). Assume that the following conditions are satisfied for all $[t, \bar{z}_{i+1}, \alpha_i - h_i(t, \bar{z}_{i+1}, l_i), \varepsilon_i] \in [0, \infty) \times D_{\bar{z}_{i+1}} \times D_{y_{i2}} \times [0, \varepsilon_0)$ for some domains $D_{\bar{z}_{i+1}} \subset R^{i+1}$ and $D_{y_{i2}} \subset R$, which contain their respective origins, where $i = 1, \ldots, n$, $\bar{z}_{n+1} = \bar{z}_n$, $D_{\bar{z}_{n+1}} = D_{\bar{z}_n}$ and $\alpha_n = u$.

($B1$) On any compact subset of $D_{\bar{z}_{i+1}} \times D_{y_{i2}}$, the functions $Q_i$, their first partial derivatives with respect to $(\bar{z}_{i+1}, \alpha_i)$ and the first partial derivative of $Q_i$ with respect to $t$ are continuous and bounded. Also $h_i(t, \bar{z}_{i+1}, l_i)$ and $(\partial Q_i / \partial \alpha_i)$ have bounded first derivatives with respect to their arguments, $(\partial Q_i / \partial \bar{z}_{i+1})$ is Lipschitz in $\bar{z}_{i+1}$, uniform in *t*.

(B2) $\partial(g_i(\bar{x}_i) y_{i1}) / \partial y_{i1}$ and $(\partial Q_i / \partial y_{i2})$ are bounded below by some positive constant for all $(t, \bar{z}_{i+1}) \in [0, \infty) \times D_{\bar{z}_{i+1}}$.

Then, the origins of (17), (18), (29), (30), (41) and (42) are exponentially stable. Besides, let $\Omega_{y_{i1}}$ be a compact subset of $\Gamma_{y_{i1}}$, where $\Gamma_{y_{i1}} \subset D_{y_{i1}}$, is the region of attraction of the autonomous system $(dy_{i1}/d\tau_{i1}) = g_i(\bar{x}_i)(-y_{i1})$. Moreover, let $\Omega_{y_{i2}}$ be a compact subset of $\Gamma_{y_{i2}}$, where $\Gamma_{y_{i2}} \subset D_{y_{i2}}$, is the region of attraction of the autonomous system $(dy_{i2}/d\tau_{i2}) = -sign\left(\frac{\partial Q_i}{\partial \alpha_i}\right) Q_i(0, \bar{z}_{i+1}, y_{i2} + h_i(0, \bar{z}_{i+1}, l_i), l_i)$ with $\bar{z}_{i+1,0} = [z_{1,0}, \ldots, z_{i+1,0}]^T$. Then, for each compact subset $\Omega_{\bar{z}_n} \subset D_{\bar{z}_n}$, there exist positive constant $\varepsilon_{i,1}{}^*$, $\varepsilon_{i,2}{}^*$ and $T > 0$ such that for all $t \geq 0$, $\bar{z}_{i+1,0} \in \Omega_{\bar{z}_{i+1}}$, $\alpha_{i,0} - h_i(0, \bar{z}_{i+1,0}, l_i) \in \Omega_{y_{i2}}$, $0 < \varepsilon_{i,1} < \varepsilon_{i,1}{}^*$ and $0 < \varepsilon_{i,2} < \varepsilon_{i,2}{}^*$, the system of (6), (14), (26) and

(38) has the unique solution $x_i(t), i = 1, \dots, n$ on $[0, \infty)$, and $x_{1,\varepsilon_1}(t) = y_d(t) + O(\varepsilon_1)$ holds uniformly for $t \in [T, \infty)$.

**Proof:** For the use of Tikhonov's theorem, it should be verified that the conditions in our theorem satisfy assumptions (A1), (A2) and (A3). First, Assumption (B1) directly implies that Assumption (A1) holds. Second, we can show easily that Assumption (A2) holds because the origins of the reduced system (16), (28) and (40) are exponentially stable equilibrium points, that is, $\|\bar{z}_n(t)\| \le \|\bar{z}_{n,0}\| e^{-w_0 t}$ for t $\ge 0$ and for some $w_0 > 0$ where $\bar{z}_{n,0} = [z_{1,0}, \dots, z_{n,0}]^T$ . From the converse Lyapunov theorem, it follows that there exists a Lyapunov function $V_c$ such that

$$w_1 \|\bar{z}_n\|^2 \le V_c(t, \bar{z}_n) \le w_2 \|\bar{z}_n\|^2 \tag{43}$$

$$\frac{\partial V_c}{\partial t}(t, \bar{z}_n) + \frac{\partial V_c}{\partial \bar{z}_n}(t, \bar{z}_n) M \bar{z}_n \le -w_3 \|\bar{z}_n\|^2 \tag{44}$$

where $w_1$ , $w_2$ , $w_3$ are positive constants and $M = diag[-m_1, \dots, -m_n]$ denotes a diagonal matrix. We note that any positive $c$ can be chosen in Assumption (A2), and so $\Omega_{\bar{z}_n} \subset \{\bar{z}_n \in D_{\bar{z}_n} | w_1(\bar{z}_n) \le \rho c, 0 < \rho < 1\}$ can be any compact subset of $D_{\bar{z}_n}$.

Finally, we show from Remark 1 that assumption (A3) holds. The exponential stability of the boundary layer system (17), (18), (29), (30), (41) and (42) can be easily obtained locally by linearization with respect to $y_{i1}$ and $y_{i2}$. Using Assumption 1 and (B2) yields

$$sign\left(\frac{\partial(g_i(\bar{x}_i) y_{i1})}{\partial y_{i1}}\right) = g_i(\bar{x}_i) > 0 \tag{45}$$

$$sign\left(\frac{\partial Q_i}{\partial y_{i2}}\right) = g_i(\bar{x}_i) > 0 \tag{46}$$

This implies that the boundary layer system has a locally exponentially stable origin. Therefore, we can apply Tikhonov's theorem. Accordingly, for each compact subset $\Omega_{\bar{z}_n} \subset D_{\bar{z}_n}$, there exist positive constant $\varepsilon_{i,1}{}^*$, $\varepsilon_{i,2}{}^*$ and $T > 0$ such that for all $t \ge 0$ , $\bar{z}_{i+1,0} \in \Omega_{\bar{z}_{i+1}}$, $\alpha_{i,0} - h_i(0, \bar{z}_{i+1,0}, l_i) \in \Omega_{y_{i2}}$ , $0 < \varepsilon_{i,1} < \varepsilon_{i,1}{}^*$ and $0 < \varepsilon_{i,2} < \varepsilon_{i,2}{}^*$ , the system of (6), (14), (26) and (38) has the unique solution $x_i(t), i = 1, \dots, n$ on $[0, \infty)$, and $x_{1,\varepsilon_1}(t) = y_d(t) + O(\varepsilon_1)$ holds uniformly for $t \in [T, \infty)$.

## IV. SIMULATION RESULTS

To validate the effectiveness of the proposed control approach, consider the following nonlinear system in the presence of both matched and unmatched uncertainties.

$$\begin{aligned} \dot{x}_1 &= 0.5\, x_1 + (1 + 0.1 x_1{}^2)(x_2 - (2 + \sin(x_1)) \\ \dot{x}_2 &= x_1 x_2 + (2 + \cos(x_1))(u - 0.3(e^{x_1} + e^{-x_2})) \end{aligned} \tag{47}$$

Where $\delta_1(x_1) = -(2 + \sin(x_1)$ is unmatched and $\delta_2(x_1, x_2) = -0.3(e^{x_1} + e^{-x_2})$ is matched uncertainty of the system.

The control object is to synthesize an adaptive control law $u$ for system (47) such that the state $x_1$ tracks the reference signal $y_d = \cos(1.5t + \pi/3) + 0.1\sin(t)$.
The initial conditions set to $x_1(0) = 1$, $x_2(0) = 1$, $u(0) = 0$, $\alpha_1(0) = 0$, $\hat{z}_1(0) = 0.5$, $\hat{z}_2(0) = 1$ and the design parameters for the proposed control system are adopted as follows: $k_1 = k_1 = 2$, $\varepsilon_{1,1} = \varepsilon_{1,2} = \varepsilon_{2,1} = \varepsilon_{2,2} = 0.01$.

Figs. 1-3 show the tracking performance, state trajectory of $x_2$ and the control input, respectively. These figures reveal that the proposed approach has the good control and tracking performance regardless matched and unmatched uncertainties. In addition, note that the states and the control input in the controlled closed-loop system are bounded.

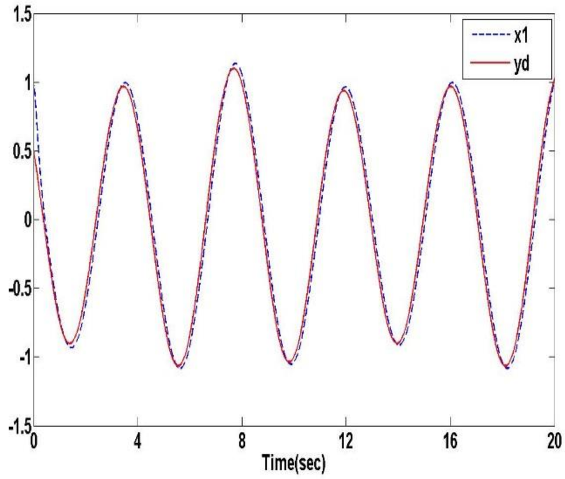


Figure 1. Tracking performance

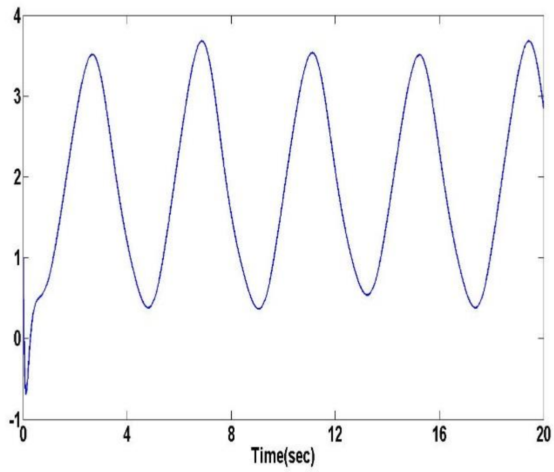


Figure 2. state trajectory of $x_2$

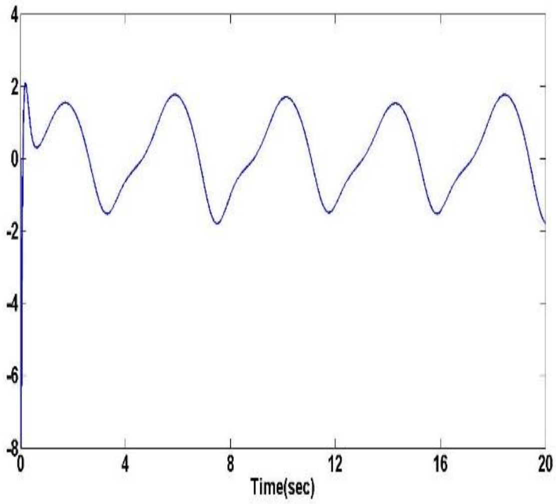


Figure 3. Control input $u$

## V. CONCLUSION

In this paper, time-scale separation has been employed in backstepping procedure for stabilization of a class of nonlinear systems in the presence of matched and unmatched uncertainties. This approach can overcome the uncertainties and also eliminates the problem of “explosion of complexity” caused by the traditional backstepping technique. Based on Tikhonov’s theorem in singular perturbation theory, the closed loop stability has been proved. The proposed controller guarantees the boundedness of all the signals in the closed-loop system, while the output of the system tracks the desired signal with bounded error.

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