Composition of Digital All-Pass Lattice Filter and Gradient Adaptive Filter for Amplitude and Delay Estimation of a Sinusoid With Unknown Frequency

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Abstract— **A composite structure for joint amplitude and time delay estimation of a delayed sinusoidal signal is proposed in this paper. The proposed method composed of a digital all-pass lattice structure and a gradient adaptive structure. The lattice structure receives the reference sinusoidal signal and provides the desirable regressor signals for the adaptive structure. Also, this structure is furnished with a frequency estimation mechanism. Applying the regressor signals, the delayed sinusoidal signal and the estimated frequency, the adaptive structure which is a gradient based filter estimates the amplitude and time-delay of the delayed sinusoid. Simulations illustrate the desirable performance of the proposed method.**

Keywords— **digital all-pass filter, lattice structure, adaptive filter, sinusoidal signals, amplitude estimation, time delay estimation.**

I. Introduction

Time delay estimation of two or more versions of a signal received at spatially separated sensors has been a major research issue in the past two decades. The problem finds many applications such as target localization, direction finding, speaker tracking, synchronization in communication receivers, biomedicine, radar and sonar ranging and speed sensing [1]-[3]. Based on the knowledge of the signal statistics, several batch estimators have been developed so far. When the characteristics of a signal is stationary a general methodology to estimate the time delay is based on locating the peak value of the cross-correlation function [4] or the generalized cross correlation function [5] of the filtered version of the two received signals. This approach can provide maximum likelihood estimation performance when the signals and noises are Gaussian distributed [6], [7]. However, the resolution of the delay estimate in these techniques is limited by the sampling period [7]. Also, due to the finite observation time, an estimate of the cross correlation is used which consequently affects the accuracy of the proposed estimator. On the other hand, several methods have been developed when the source signal is deterministic, specifically for a pure sinusoid that commonly occurs in radar, sonar and digital communications applications. Some methods are quadrature delay estimator (QDE) [8], [9], estimators based on phase difference of the discrete-time Fourier transforms of the received signals [10]-[12] and estimators based on a combination of cross correlation and autocorrelation [13].

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If the delay is time-varying due to relative source/recevier motion, or if the delay estimate needs to be determined at each new input sample, online or adaptive techniques are usually preferred. For a special case, that is when both the received signals are sinusoids with unity amplitude, [6] provides a method for the online estimation of delay. In practice, the relative amplitude of the signals is also unknown and time varying [7]. In [7] a modification to [6] has been presented which results in an adaptive filter that estimates both delay and amplitude of a delayed sinusoid. In order to implement the algorithms proposed in [6] and [7], it is required to choose the sampling rate from a specific set of permissible values which requires the knowledge of the input signals frequency. However, the frequency of signals may be unknown and time-varying. This brings about the necessity of estimating the frequency.

The main contribution of this paper is the development of a method for the online estimation of time delay and relative amplitude of one of two received sinusoidal signals vis-a-vis the other. The proposed method comprises an all-pass filter and an adaptive structure. The all-pass filter which has a lattice structure receives the reference sinusoidal signal and generates the suitable regressor signals for the adaptive structure. Also, an update law is employed to tune the notch frequency of the filter on the input sinusoidal signal. The adaptive structure which is a gradient based adaptive filter estimates the amplitude and time-delay by using the signals generated by the lattice structure and minimizing a suitable error function. The stability analysis is also carried out to ensure convergence of the estimated values to the true values. Adaptive nature of the proposed method enables tracking of slow variations of time-delay and amplitude. The structural simplicity of the proposed estimator makes it suitable for digital implementation both in hardware and software environments.

II. Adaptive Estimation of Delay and Amplitude

A. Introduction

Consider the following sinusoidal signals received at the two sensors,

$$\begin{aligned} u_a(t) &= \sin\left(\omega_0 t + \delta_0\right) \\ u_b(t) &= a_0 \sin\left(\omega_0 (t - d_0) + \delta_0\right) \end{aligned} \tag{1}$$

where ω_0 and δ_0 are respectively the angular frequency and initial phase. d_0 represents the unknown difference in arrival times at two receivers. a_0 is a gain factor associated with delayed sinusoid and is assumed to be unknown. Our aim is to find an adaptive algorithm for direct estimation of a_0 and d_0. In order to avoid ambiguities, it is assumed that the net phase shift $\omega_0 d_0$ lies within the interval $[0, 2\pi)$.

Assume that the signals u_a and u_b are sampled with a sampling period of T to generate the sequences

$$\begin{aligned} u_a[n] &= u_a(nT) = \sin(\omega_0 T n + \delta_0) \\ u_b[n] &= u_b(nT) = a_0 \sin(\omega_0 T n + \delta_0 - \omega_0 d_0) \end{aligned} \tag{2}$$

It is then possible to express $u_b[n]$ as

$$u_b[n] \quad = \quad a_0 \mathbf{x}^{\mathsf{T}}[n] \mathbf{w}_0 \tag{3}$$

where $\mathbf{w}_0 = \big(-\sin(\omega_0 d_0),\ \cos(\omega_0 d_0) \big)^{\mathsf{T}}$ and $\mathbf{x}[n] = \big(x_1[n],\ x_2[n] \big)^{\mathsf{T}} = \big(\cos(\omega_0 T n + \delta_0),\ \sin(\omega_0 T n + \delta_0) \big)^{\mathsf{T}}$. The idea is to treat (3) as a system identification model with regressor signals $x_1[n]$ and $x_2[n]$. This idea requires a mechanism to produce regressor signals. In [6] and [7], the regressor signals have been generated by choosing the sampling frequency from a specific set of permissible values based on the input signals frequency, which restricts the sampling frequency to specific values. On the other hand, there exist structures capable of generating the orthogonal signals (sin / cos) from an input sinusoidal signal which can be used for this purpose. The digital all pass lattice structure is one of such structures that is reviewed in the next section.

B. Review of Lattice Structure

The digital all pass filter is a computationally efficient signal processing building block which is quite useful for many signal processing applications [14]. A suitable choice of all-pass structure is the planar rotation lattice filter (Fig. 1), as the structure is theoretically stable and numerically well behaved in time varying environments [15]. Tunable planar rotations are best implemented by a sequence of CORDIC rotations [16]. Each rotation angle $\theta_k,\ k = 1, 2$ is directly controlled via its own register in CORDIC. With Ω denoting the notch frequency of (notch) filter $F(z) = \frac{1}{2}(1 + A(z))$ and B the 3-dB attenuation bandwidth, one can show that [15]

$$\theta_1 = \Omega - \frac{\pi}{2},\ \Omega \in [0, \pi], \qquad \sin\theta_2 = \frac{1 - \tan(B/2)}{1 + \tan(B/2)}$$

The update law

$$\theta_1[n+1] \quad = \quad \theta_1[n] - \mu_0 e_0[n] x_1[n] \tag{4}$$

is proposed to tune the notch frequency parameter θ_1 [15]. In this update law, the error signal $e_0[n]$ is the output of the notch filter $F(z)$ and the regressor signal $x_1[n]$ is available from the lattice filter (see Fig. 1). In the case of a single sinusoid $u[n] = u_a[n] = \sin\big(\omega_0 T n + \delta_0\big)$ applied to the lattice structure, the update law (4) is shown to be asymptotically stable in the sense that $\sin\theta_1[n] \to -\cos\omega_0 T$ as $n \to \infty$. This means that $\theta_1[n] \to \omega_0 T - \pi/2$ as $n \to \infty$, and therefore, the input signal frequency can be estimated from $\frac{\theta_1[n] + \pi/2}{T}$.

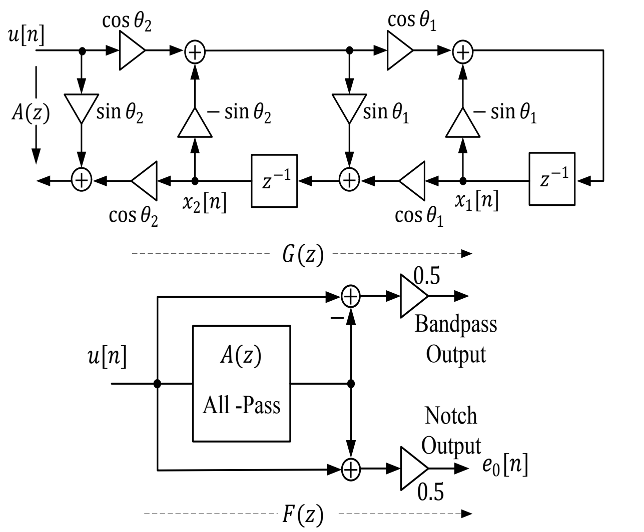

Fig. 1. Digital all pass lattice filter (top) and two lossless filter-bank (bottom) [15].

In this case, it can be observed that the lattice structure has the steady state solution of

$$\begin{pmatrix} \bar{x}_1[n] \\ \bar{x}_2[n] \end{pmatrix} = \frac{-1}{\sqrt{\tan(B/2)}} \begin{pmatrix} \cos\big(\omega_0 T n + \delta_0\big) \\ \sin\big(\omega_0 T n + \delta_0\big) \end{pmatrix} \tag{5}$$

Therefore, with a gain scaling of $-\sqrt{\tan(B/2)}$, we have signals $\big(x_1[n],\ x_2[n]\big) = \big(\cos(\omega_0 T n + \delta_0),\ \sin(\omega_0 T n + \delta_0)\big)$ which are suitable as the regressor signals.

C. Amplitude and Time-Delay Estimation

To derive the governing equations of the estimator straightforward, it is assumed that the lattice structure reaches to its steady state and provides the regressor signals $\big(x_1[n],\ x_2[n]\big)^{\mathsf{T}}$ and the frequency ω_0.

Using the relation (3), the estimate of $u_b[n]$ denoted by $\hat{u}_b[n]$, can be represented as

$$\hat{u}_b[n] \quad = \quad a[n] \mathbf{x}^{\mathsf{T}}[n] \mathbf{w}[n] \tag{6}$$

where $\mathbf{w}[n] = \big(-\sin(\omega_0 d[n]),\ \cos(\omega_0 d[n]) \big)^{\mathsf{T}}$. Define the instantaneous error between $u_b[n]$ and its estimate $\hat{u}_b[n]$ as

$$\begin{aligned} e[n] &= u_b[n] - \hat{u}_b[n] \\ &= u_b[n] - a[n] \mathbf{x}^{\mathsf{T}}[n] \mathbf{w}[n], \end{aligned}$$

We are looking for those values of $\phi[n] = (a[n], d[n])^{\mathsf{T}}$ that minimize a function of $e[n]$. A suitable and well-known function is the instantaneous square error as

$$\begin{aligned} J(n, \phi[n]) &= \tfrac{1}{2} e^2[n] \\ &= \tfrac{1}{2} \big(u_b[n] - a[n] \mathbf{x}^{\mathsf{T}}[n] \mathbf{w}[n] \big)^2 \end{aligned}$$

The gradient descent (GD) method provides a method of adjusting the unknown parameter vector ϕ so that the cost function J tends to its minimum point. In this method, any unknown parameter moves to the opposite direction of the variations of the function J with respect to that parameter.

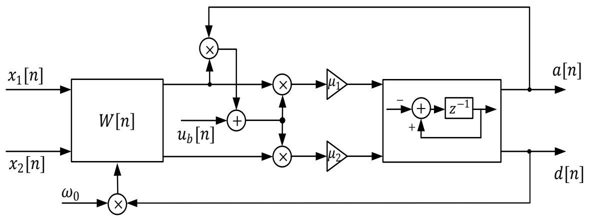

Fig. 2. Proposed adaptive structure for estimation of amplitude and time delay.

If μ is a diagonal matrix with positive diagonal elements, then the GD method is formulated as

$$\begin{aligned} \phi[n+1] &= \phi[n] - \mu \nabla J \\ &= \phi[n] - \mu e[n] \nabla e[n] \end{aligned} \tag{7}$$

where

$$\nabla e[n] = \Big(\frac{\partial e[n]}{\partial a[n]}, \frac{\partial e[n]}{\partial d[n]}\Big)^{\mathsf{T}}$$

Following the GD method for the parameter vector $\phi = (a, d)^{\mathsf{T}}$, one will obtain the following difference equation

$$\phi[n+1] = \phi[n] - \mu \Gamma[n] \mathbf{x}[n] e[n] \tag{8}$$

where

$$\Gamma[n] = \begin{pmatrix} 1 & 0 \\ 0 & \omega_0 a[n] \end{pmatrix} \begin{pmatrix} \sin(\omega_0 d[n]) & -\cos(\omega_0 d[n]) \\ \cos(\omega_0 d[n]) & \sin(\omega_0 d[n]) \end{pmatrix}$$

and $\mu = diag(\mu_1, \mu_2)$. $\mu_i, i = 1, 2$ control the convergence rate as well as the stability of the adaptive filter as can be seen later.

The term $\omega_0 a[n]$ may be absorbed into μ_2 without any degrading effect on the basic features of the filter. Based on this modification, the governing equations of the proposed filter for amplitude and time-delay estimation of a delayed sinusoid is

$$\phi[n+1] = \phi[n] - \mu \Gamma[n] \mathbf{x}[n] \big(u_b[n] - a[n] \mathbf{x}^{\mathsf{T}}[n] \mathbf{w}[n]\big) \tag{9}$$

where $\Gamma[n]$ is redefined as

$$\Gamma[n] = \begin{pmatrix} \sin(\omega_0 d[n]) & -\cos(\omega_0 d[n]) \\ \cos(\omega_0 d[n]) & \sin(\omega_0 d[n]) \end{pmatrix}$$

Fig. 2 shows a block diagram representation of the proposed adaptive filter. The main computation in this block diagram is a rotation on the vector $\mathbf{x}[n]$ by the angle $\omega_0 d[n]$. This rotation can be carried out by a sequence of CORDIC rotations [16] and, it can efficiently be implemented on a pipelined array of CORDIC processors [7]. Also, this rotation evaluates $-\mathbf{w}^{\mathsf{T}}[n]\mathbf{x}[n]$ (top output) which after multiplication by $a[n]$ provides the estimate of the delayed signal with a minus sign, i.e. $-\hat{u}_b[n]$.

D. Convergence Analysis

The GD method is guaranteed to converge to minimum solution if the cost function is globally quadratic in parameters. Otherwise, i.e. if the form of the cost function is not quadratic, like the present situation, a mathematical proof should be given to guarantee the convergence of the solutions of the GD method to the minimum point of the cost function. Using the relation $u_b[n] = a_0 \mathbf{x}^{\mathsf{T}}[n] \mathbf{w}[n]$, the equation (9) can be expressed as

$$\phi[n+1] = \phi[n] - \mu \Gamma[n] \mathbf{x}[n] \mathbf{x}^{\mathsf{T}}[n] (a_0 \mathbf{w}_0 - a[n] \mathbf{w}[n]) \tag{10}$$

where,

$$\mathbf{x}[n]\mathbf{x}^{\mathsf{T}}[n] = \begin{pmatrix} \cos^2(\omega_0 T n + \delta_0) & \frac{1}{2}\sin 2(\omega_0 T n + \delta_0) \\ \frac{1}{2}\sin 2(\omega_0 T n + \delta_0) & \cos^2(\omega_0 T n + \delta_0) \end{pmatrix}$$

With the definition of $\mu_i = \epsilon \bar{\mu}_i, i = 1, 2$, where ϵ is a small positive parameter, equation set (10) is in the standard form to which the discrete averaging theorem can be applied (See Appendix). The corresponding average system is[1]

$$\overline{\phi}[n+1] = \overline{\phi}[n] - \tfrac{1}{2} \mu \overline{\Gamma}[n] (a_0 \mathbf{w}_0 - \overline{a}[n] \overline{\mathbf{w}}[n]) \tag{11}$$

After doing calculations, we have

$$\begin{aligned} \overline{a}[n+1] &= (1 - \tfrac{\mu_1}{2}) \overline{a}[n] + \tfrac{\mu_1}{2} a_0 \cos\big(\omega_0 (\overline{d}[n] - d_0)\big) \\ \overline{d}[n+1] &= \overline{d}[n] - \tfrac{\mu_2}{2} a_0 \sin\big(\omega_0 (\overline{d}[n] - d_0)\big) \end{aligned} \tag{12}$$

The nonlinear difference equations (12) have a fixed point at $(\bar{a}, \bar{d}) = (a_0, d_0)$. Linearizing these equations around the fixed point results in,

$$\overline{e}[n+1] = \begin{pmatrix} 1 - \frac{\mu_1}{2} & 0 \\ 0 & 1 - \frac{\mu_2}{2} a_0 \omega_0 \end{pmatrix} \overline{e}[n] \tag{13}$$

where $\overline{e} = (\overline{a}, \overline{d})^{\mathsf{T}} - (a_0, d_0)^{\mathsf{T}}$. The linear system (13) is asymptotically stable if and only if $|1 - \frac{\mu_1}{2}| < 1$, $|1 - \frac{\mu_2}{2} a_0 \omega_0| < 1$ or equivalently $0 < \mu_1 < 4$, $0 < \mu_2 < \frac{4}{a_0 \omega_0}$. This implies that for this selection of step sizes $\mu_i, i = 1, 2$, $(a_0, d_0)^{\mathsf{T}}$ is a locally asymptotically stable fixed point for dynamical system (12). $(a_0, d_0)^{\mathsf{T}}$ is also a fixed point for (10), therefore, based on the averaging theory the update laws (10) (and consequently (9)) are locally asymptotically stable, in the sense that $(a[n], d[n]) \to (a_0, d_0)$ as $n \to \infty$.

E. Comparison With Method of [7]

In comparison with the method of [7], the features of the proposed method are as follows:

- To produce the regressor signals, the idea in [7] is to select the sampling frequency as $f_s = 4 r f_0, r \in \{1, 2, ...\}$ which restricts the sampling frequency to specific values based on the input signals frequency. The input signal frequency may be time-varying or even unknown. Frequency estimation capability of the proposed method makes it applicable in various sampling frequency independent of input signals frequency.
- The GD method has been used to derive the governing equations of the adaptive part of both methods, but absorbing $\omega_0 a[n]$ into μ_2 makes the adaptive part of our proposed method slightly simpler.

[1]Note that the average of $\mathbf{x}[n]\mathbf{x}^{\mathsf{T}}[n]$ is $\frac{1}{2} I_2$ where I_2 is an 2×2 identity matrix.

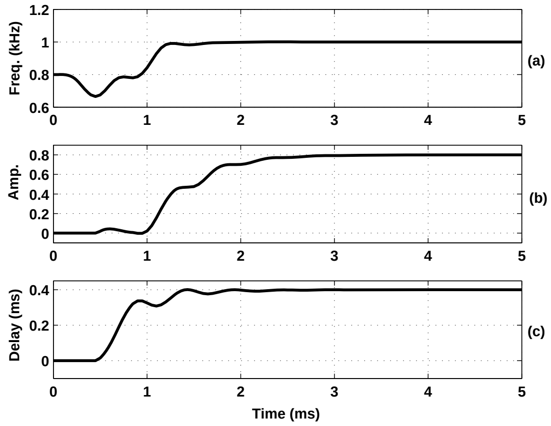

Fig. 3. Initiatory performance of the proposed algorithm in the fixed parameters conditions, **a)** estimated frequency, **b)** estimated amplitude **c)** estimated time-delay.

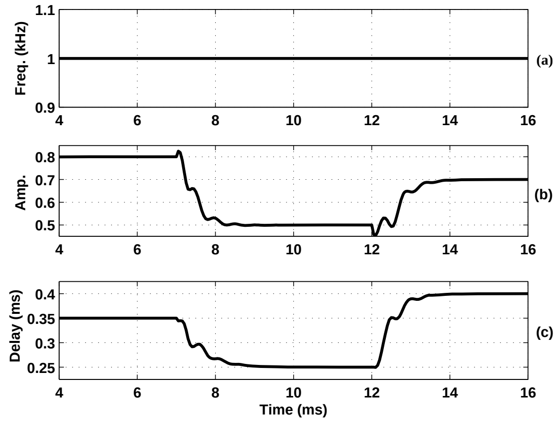

Fig. 4. Performance of the proposed algorithm in tracking step changes in amplitude and time delay, **a)** estimated frequency, **b)** estimated amplitude, **c)** estimated time delay.

• Basically, there is no difference between two methods in noise analysis (bias and variance). Except that in the proposed method the reference signal $u_a[n]$ passes through a digital all-pass filter and therefore, the filtered noise is added to the regressor signals $x_1[n]$ and $x_2[n]$. However, simulation results show that noise immunity feature of the proposed method is desirable and competes with that of the method proposed in [7].

III. Performance Study

In this section a number of simulation examples are given to evaluate the performance of the proposed method. All simulations are conducted in MATLAB/Simulink environment. In simulation examples, $u_a(t) = \sin(2\pi \times 1000t)$ is assumed as a reference signal and the sampling frequency is chosen as $f_s = 20$kHz. The parameters of the lattice structure are $B = \frac{\pi}{6}$ and $\mu_0 = 0.04$. The step sizes for the adaptive filter are selected as $\mu_1 = 0.2$ and $\mu_2 = \frac{1}{5000\pi}$.

In the first example the initiatory performance of the proposed algorithm is evaluated. For this, a sinusoidal signal of amplitude $a_0 = 0.8$ and time delay $d_0 = 0.4$ ms is assumed for $u_b(t)$. The estimated values of frequency, amplitude and time-delay are shown in Fig. 3. It is observed that the estimated parameters converge to their nominal values in about two cycles of the input signals.

In another simulation the capability of the proposed filter in tracking step changes of the amplitude and time delay is studied. It is assumed that the initial values of amplitude and delay are specified as 0.8 and 0.35 ms and the filter is on its nominal condition. Two simultaneous step changes in amplitude from 0.8 to 0.5 and from 0.5 to 0.7, and in time delay from 0.35 ms to 0.25 ms and from 0.25 ms to 0.4 ms occur at $t = 7$ ms and $t = 12$ ms, respectively. The estimated values of frequency, amplitude and time delay are shown in Fig. 4. It can be seen that the proposed algorithm faithfully tracks step changes. Obviously, these step changes have no effect on the estimated frequency

Fig. 5 illustrates the mean of the estimated parameters for 500 independent runs of the proposed method. The desirable performance of the proposed estimator in a noisy environment can be seen from Fig. 5. Also, it can be seen that while both inputs have the same signal-to-noise (SNR) ratios ($\mathrm{SNR}_a = \mathrm{SNR}_b = 17$dB), the variances of amplitude and time-delay are less than variance of frequency. This is because of passing $u_a[n]$ through all-pass filter which causes a filtered noise is added to the regressor signals.

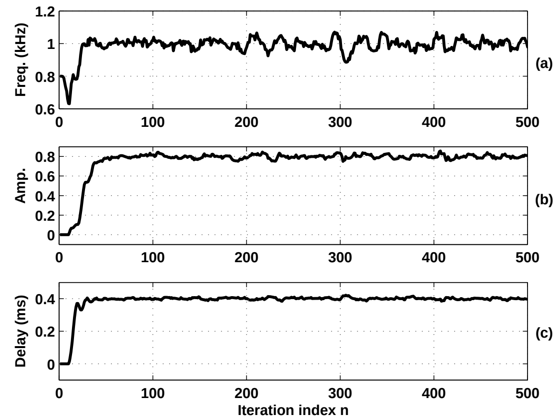

Fig. 5. Illustration of the estimated parameters based on the mean of 500 independent runs of the proposed method for $\mathrm{SNR}_a = \mathrm{SNR}_b = 17$dB, **a)** frequency, **b)** amplitude, **c)** time-delay.

IV. Conclusions

Mathematical formulation and performance evaluation of an algorithm for amplitude and time delay estimation of a delayed sinusoidal signal with unknown frequency is presented. The proposed algorithm comprises a digital all pass filter and a gradient based adaptive filter. The proposed algorithm is applicable at various sampling frequency independent of the frequency of the input signal. Both portions of the algorithm are simple and well suited for implementation on CORDIC processors. Simulation studies verify the desirable performance of the proposed algorithm.

Appendix

A simplified version of the discrete averaging theorem can be stated as follows [17]. Consider the difference equation of the form

$$\phi[n+1] = \phi[n] + \epsilon f(n, \phi[n], \epsilon)$$

where ϵ is a small positive parameter. Define the average system as

$$\overline{\phi}[n+1] = \overline{\phi}[n] + \epsilon \overline{f}(\overline{\phi}[n])$$

where

$$\overline{f}(\overline{\phi}) = \lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N} f(n, \overline{\phi}, 0).$$

Assume that both systems have the same fixed point. Then, the stability type of fixed point of the original system can be inferred from that of the fixed point of the average system.

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