Convergence Improvement in Finite Difference Solution Using MC and DC Methods for Magnetic Field Analysis

Mehdi Salehi[1], Hossein Torkaman[2], Ali Safdari[3]

[1] Faculty of Electrical Engineering, University of Victoria, Victoria, Canada.
[2] Faculty of Electrical Engineering, Shahid Beheshti University, A.C., Tehran, Iran.
[3] Faculty of Mathematical Science, Allameh Tabatabaei University, Tehran, Iran.
msalehi@uvic.ca, h_torkaman@sbu.ac.ir, asafdari@atu.ac.ir

Abstract**—**Among the numerical methods used in the electromagnetic modeling and simulation of electrical systems, the iterative method is included. In this paper, different techniques are employed to a classical Gauss-Seidel Algorithm. It used to improve accuracy and convergence of the solutions for a common partial differential equation in finite difference method. The first method is named Double Convergence Method in which combinations of two Iteration Methods with different initial points are utilized. The second method is called Multi-level Convergence Method where, a mixture of multi Iteration Method with initial values obtained from previous processes using first, second, third order polynomial for the next round of iteration. The convergence time and accuracy of both methods are evaluated and compared using classical Gauss-Seidel Algorithm by solving various one dimensional partial differential equations. The aim of this paper is to reduce the number of iterations of this method in order to reduce the computing time and to improve the convergence speed.

KeyWords**— Convergence, Finite Difference Method, Iteration Method, Partial Differential Equations, Electromagnetics Fields.**

I. INTRODUCTION

In general, every phenomenon in the world, whether mechanical such as heat or electrical such as magnetic or electric fields, can be described with the help of the laws of physics, which results in mathematical formulations of physical problem in terms of partial differential, or integral equations relating various quantities of interest. Even though the derivation of the governing equations for most of the problems is not very hard, but finding analytical solutions in different shapes and geometries is a difficult task. In such cases, approximate methods of analysis offer alternative means of finding solutions. Among these, the finite difference scheme, the Finite Element techniques, and the boundary element methods are the most frequently used in literature. In finite difference approximation of a differential equation, the derivatives are replaced by the difference quotients found by the Taylor's series expansion and the approximate solution is found at discrete mesh points of the domain [1-6]. The linear and non-linear Laplace's or Poisson's equation has been investigated in literature for different applications [7-11].

The electromagnetic field analysis in electrical systems utilizing the finite difference (FD) and finite element (FE) methods gives fairly good agreement with experimental results. For example, the electrical machines [12-15] and other devices [16-18] have been analyzed utilizing the FE method and also the electromagnetics fields have been analyzed utilizing FD method [19-22].

One of the most important techniques in solving the partial differential equations by using FDM is the iteration method (IM). In this method the finite deference equation is solved based on initial guesses for unknown parameters. Then the new results compare with previous results. These steps are repeated to achieve a proper tolerance. Gauss-Seidel, Jacobi, and Newton-Raphson are some of the most famous techniques in IM [1, 23]. Although, their implementation are so easy they suffer from long-time solution and divergence. Several modifications are applied to these techniques in the recent literatures [24].

In this paper, two new techniques are presented to overcome the above mentioned drawbacks. In the first method the finite difference equation is solved by minimum number of grid points and then using these values an equation is developed. This equation provides suitable initial guesses for the desired number of grid points in the next round of iteration. Therefore, this technique is named double convergence method (DCM) since two rounds of iteration are processed. The second method is a combination of several Iteration Methods in which the initial values are obtained from previous processes. Again in every round of iteration either first, or second, or third degree equation is developed. These equations are used to produce the suitable initial guesses for the next round of iteration with more grid points. This procedure is repeated as many times until the desired number of points are achieve. This technique is named Multi-level Convergence Method (MCM).

This paper is organized as follows: in section II the process of each methods are presented. The implementation of the methods in solving the partial differential equations is discussed in section III. Finally, section IV concludes the paper.

II. DESCRIPTION OF THE METHODS

Iteration method is a common technique in different software and applications. In this method the partial differential equations are solved by initial guesses for unknown parameters. Then new results are compared with previous results to achieve the desired tolerance. It is noteworthy that initial guesses plays significant role in convergence of this technique. The improper initial data selection leads to either long-time solution or divergence. The proposed methods are a linear system solver, using Gauss-Seidel iteration. These methods can be applied to general linear system, which are useful for Finite Element method since FEM usually needs a linear system solver.

Different approaches are reported in [24-27] to improve these issues. However these techniques are not applicable and general for variety of equations. The implementation and description of the two methods are explained in the following manner.

1) Double Convergence Method (DCM)

In this method, the problem is solved utilizing the minimum number of grid points in the first step in order to come up with appropriate values for the grid points as well as a fast and speedy convergence. In this stage, if the problem is solved for *m* grid points, then a (m-1) order equation can be developed using least square method to fits these points. After that, the second round of iteration starts for n points. The initial values for the n points are obtained using the (m-1) order equation developed in pervious step.

Considering m to be equal to 2 or 3, then the solution space is estimated by a linear equation or a quadratic equation which is well-suited for one dimensional Laplace's or Poisson's equation.

2) Multi-level convergence method (mcm)

In this method, the problem is solved utilizing the m grid points in the first step. The grid points are doubled, and depending on m which is an odd integer number, a first, second, or third degree polynomial using least square method is developed in this study. Hence, for each (m+1)/2 data points, a (m-1)/2 order curve is fitted through the grid point values. Next, the number of points is increased by (2m-1) in each step of iteration, and the problem is solved again for the new number grid points using the curves obtained in the previous iteration to estimate the initial guesses. Finally, this procedure is repeated as many times necessary to achieve the desired number of grid points.

If m is considered to be equal 3, then the solution space is estimated by piecewise linear equations which are known as multi-grid method which has been reported in [28-30].

It is noteworthy to mention that by increasing the value of m the accuracy of solution as well as the time of solution goes up. For example, assuming m=5, the solution space is estimated by quadratic equations which improves the accuracy at the expense of the solution time.

III. Comparative Study of the Methods

In order to evaluate the proposed methods, the following one-dimensional partial differential equation in finite area ($0 \le x \le 1$) is considered:

$$\begin{cases} \dfrac{\partial^2 \mathrm{T}}{\partial x^2} = \alpha x + \beta \sin(\omega x) + \gamma \\ T(0) = k1, \quad T(1) = k2 \end{cases} \qquad (1)$$

where, $\alpha, \beta, \omega, \gamma$, k_1 and k_2 are constant.

Different values of constant parameters will produce three significant equations which are solved by the previously mentioned methods.

1) Case I: Laplace's equation

In this case, $\alpha, \beta, and\ \gamma$ are equal to zero. Therefore, the equation with two boundary conditions can be rewritten as;

$$\begin{cases} \dfrac{\partial^2 \mathrm{T}}{\partial x^2} = 0 \\ T(0) = 0, \quad T(1) = 100 \end{cases} \qquad (2)$$

The analytical solution of (2) is

$$T(x) = 100x \quad for \ \ 0 \le x \le 1 \qquad (3)$$

Fig. 1 and 2 show the solution time and the number of iterations versus the degree of approximation in DCM and MCM techniques, respectively.

Utilizing Gauss-Seidel method, the number of iterations as well as the solution time obtained for *n=1025* grid points are 1780152 times and 100 second, respectively. As depicted in Fig. 1, the time takes for DCM and MCM techniques are in [0.0063-0.0098] and [0.035-0.039] intervals respectively, which shows a much faster convergence time.

The analytical solution for equation (3) is linear, hence the error for the solution in DCM and MCM techniques are almost the same and much more accurate than applying ordinary Gauss-Seidel method. This fact is also shown in Table I.

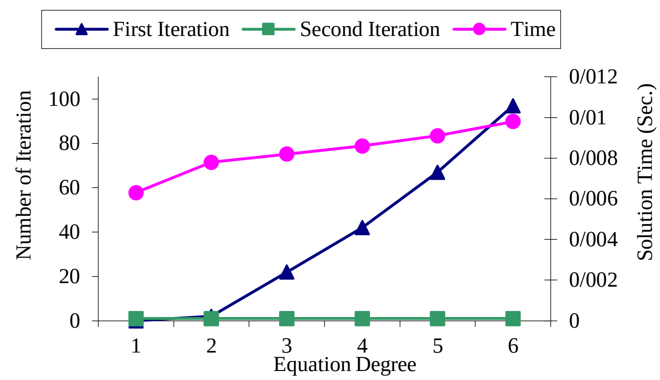

Fig. 1. The solution time versus equation degree for DC method in case I.

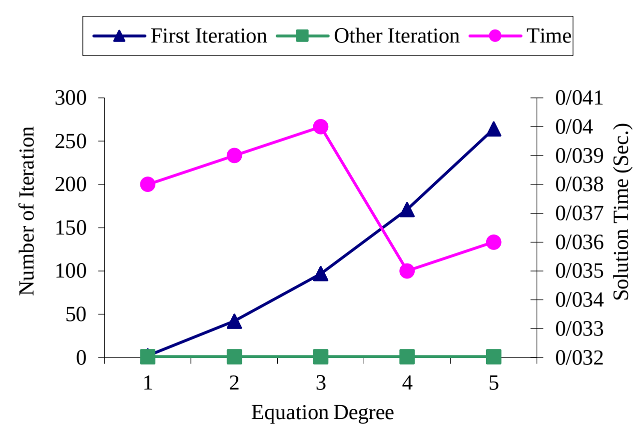

Fig. 2. The solution time versus equation degree for MC method in case I.

TABLE I

NUMBER OF ITERATIONS, NUMBER OF STARTING GRID POINTS, DEGREES OF POLYNOMIAL, ERRORS AND SOLUTION TIME OF EQUATION FOR CASE I.

Methods	***Number of Iterations***							***Number of Starting grid Points (m)***	***Degrees of Polynomial***	***Errors***	***Solution Time(sec.)***
Gauss-Seidel	**1780152**							**1025**	**--**	**2.1e-6**	**100**
DCM	**0**	**1**						**2**	**1**	**0**	**0.0063**
	2	**1**						**3**	**2**	**0**	**0.0078**
	22	**1**						**4**	**3**	**3.4e-12**	**0.0082**
	42	**1**						**5**	**4**	**1.3e-11**	**0.0086**
	67	**1**						**6**	**5**	**2.5e-11**	**0.0091**
	97	**1**						**7**	**6**	**4e-11**	**0.0098**
MCM	**2**	**1**	**1**	**1**	**1**	**1**	**1**	**3**	**1**	**0**	**0.038**
	42	**1**	**1**	**1**	**1**	**1**	**1**	**5**	**2**	**1e-11**	**0.039**
	97	**1**	**1**	**1**	**1**	**1**	**1**	**7**	**3**	**3.6e-11**	**0.040**
	171	**1**	**1**	**1**	**1**	**1**	**1**	**9**	**4**	**8.2e-11**	**0.035**
	264	**1**	**1**	**1**	**1**	**1**	**1**	**11**	**5**	**1.4e-10**	**0.036**

2) Case II: Poisson's equation

In this case α, β are equal to zero and γ is (-200). Therefore, the equation with two boundary conditions can be rewritten as;

$$\begin{cases} \dfrac{\partial^2 \mathrm{T}}{\partial x^2} = -200 \\ T(0) = 0, \quad T(1) = 10 \end{cases} \tag{4}$$

The analytical solution for equation (4) is

$$T(x) = -100x^2 + 110x \quad for \;\; 0 \le x \le 1 \tag{5}$$

Fig. 3 and Fig. 4 show the solution time and number of iteration versus the degree of approximation in DCM and MCM techniques for case II, respectively.

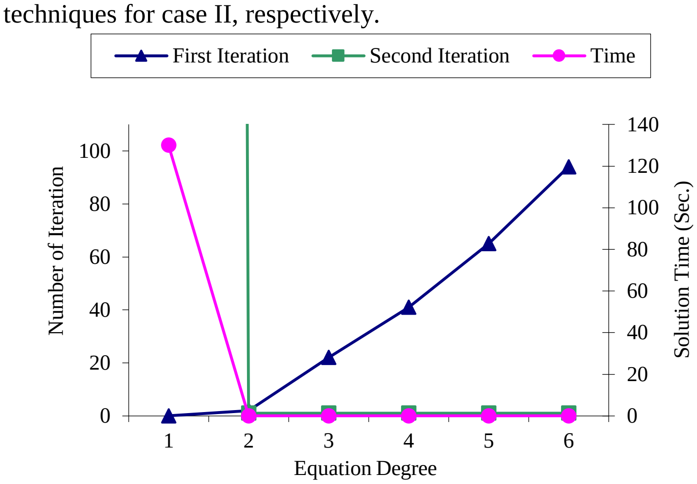

Fig. 3. The solution time versus equation degree for DC method in case II.

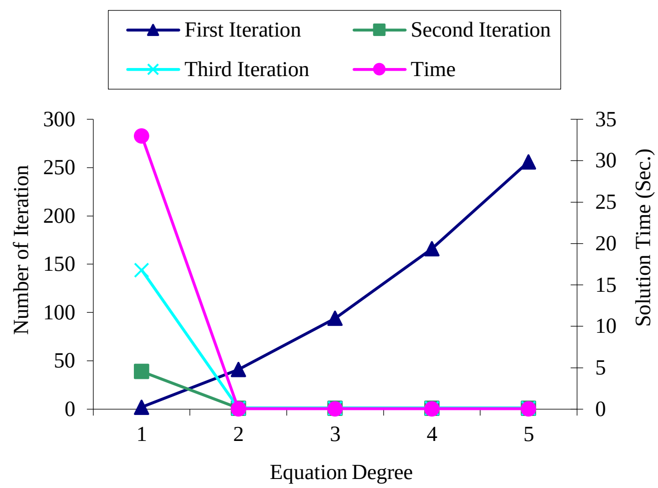

Fig. 4. The solution time versus equation degree for MC method in case II.

Utilizing Gauss-Seidel method, the number of iterations as well as the solution time obtained for *n=1025* grid points are 1715580 times and 133 second, respectively. As depicted in Fig. 1, the time takes for DCM and MCM techniques are in [0.0087-130] and [0.032-33] intervals respectively, which shows much faster convergence time. Comparing the results obtained by these methods with the analytical solution show the accuracy of MCM and DCM are equal but better than applying ordinary Gauss-Seidel method.

As given in (5), the analytical solution for this equation is quadratic, and also the error of solution in DCM (for degree>=2) and MCM (for degree>=1) are small and the solution times are minimum as shown in Table II.

TABLE II

NUMBER OF ITERATIONS, NUMBER OF STARTING GRID POINTS, DEGREES OF POLYNOMIAL, ERRORS AND SOLUTION TIME OF EQUATION FOR CASE II.

Methods	***Number of Iterations***							***Number of Starting grid Points (m)***	***Degrees of Polynomial***	***Errors***	***Solution Time(sec.)***
Gauss-Seidel	**1715580**							**1025**	**--**	**1e-4**	**133**
DCM	**0**	**1692099**						**2**	**1**	**1e-4**	**130**
	2	**1**						**3**	**2**	**0**	**0.0087**
	22	**1**						**4**	**3**	**8.8e-11**	**0.0090**
	41	**1**						**5**	**4**	**6e-10**	**0.0095**
	65	**1**						**6**	**5**	**1.3e-9**	**0.0098**
	94	**1**						**7**	**6**	**2.1e-9**	**0.010**
MCM	**2**	**39**	**144**	**509**	**1748**	**5838**	**18740**	**3**	**1**	**2.1e-3**	**33**
	41	**1**	**1**	**1**	**1**	**1**	**1**	**5**	**2**	**4.8e-10**	**0.039**
	94	**1**	**1**	**1**	**1**	**1**	**1**	**7**	**3**	**1.9e-9**	**0.041**
	166	**1**	**1**	**1**	**1**	**1**	**1**	**9**	**4**	**4.1e-9**	**0.032**
	256	**1**	**1**	**1**	**1**	**1**	**1**	**11**	**5**	**7.4e-9**	**0.036**

3) Case III: Sinusoidal equation

In this case α is equal to zero, β is $(-(5\pi)^2)$ and γ is equal to -10. Therefore, the equation with two boundary conditions can be rewritten as;

$$\begin{cases} \dfrac{\partial^2 \mathrm{T}}{\partial x^2} = -10 - (5\pi)^2 \sin(5\pi x) \\ T(0) = 0, \quad T(1) = 1 \end{cases} \qquad (6)$$

The analytical solution for equation (6) can be written as;

$$T(x) = -5x^2 + 6x + \sin(5\pi x) \quad for \quad 0 \le x \le 1 \qquad (7)$$

Fig. 5 and Fig. 6 show the solution time versus the degree of approximation for different methods in this case.

Employing Gauss-Seidel method, the number of iterations as well as the solution time obtained for *n=1025* grid points are 1868068 times and 258 second, respectively. As depicted in Fig. 1, the time takes for DCM and MCM techniques are in [227-253] and [13.5-79] intervals respectively, which shows much faster convergence time.

The exact solution of equation (6) is a sinusoidal function, which can be represented by a unique power series expansion as a function of *x*. Hence, by increasing the value of *m* in DCM the convergence time will also decrease since the initial guesses move toward the analytical solution of the equation. The convergence time in MCM is smaller than the other methods. In addition, in MCM the accuracy of solution will improve as the value of *m* goes up.

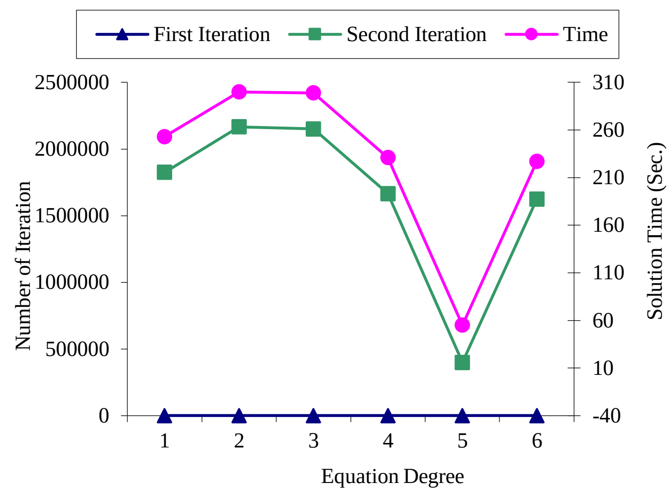

Fig. 5. The solution time versus equation degree for DC method in case III.

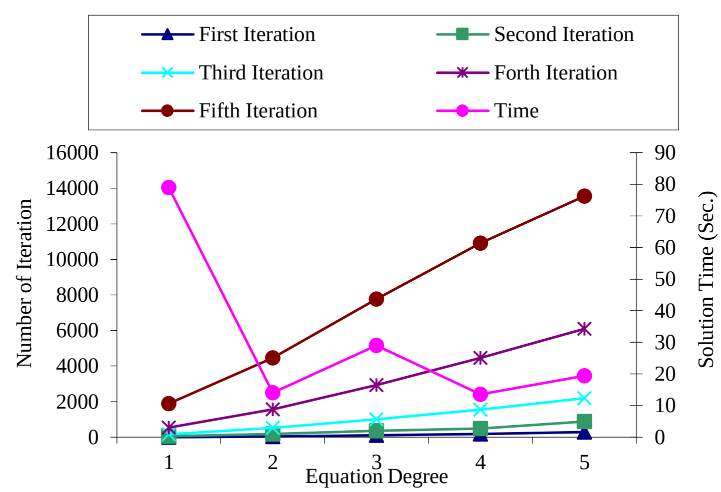

Fig. 6. The solution time versus equation degree for MC method in case III

Finally, for case II the solution time versus number of iteration for different degrees of the developed polynomials are illustrated in Fig. 7 and Fig. 8 by DC and MC methods, respectively. Simulation results show good agreements between these two methods. The solution time in DCM is smaller than that of MCM. The reason for that is because the analytical solution for equation (5) is same as the fitted curve.

Finally, number of iterations, number of starting grid points, degrees of polynomial, errors and solution time for different cases are shown in Table I, II, and III respectively.

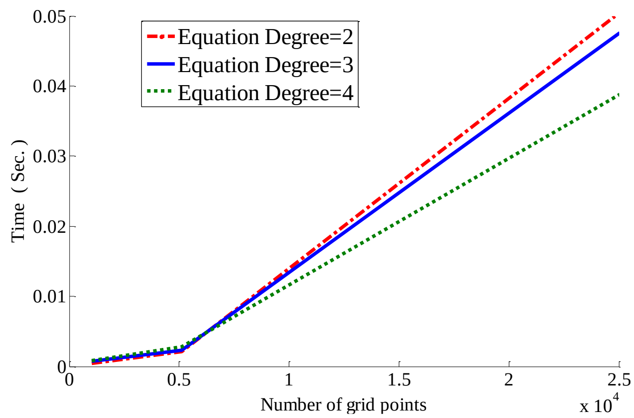

Fig. 7. The solution time versus number of grid points for different equation degree for DCM in case II.

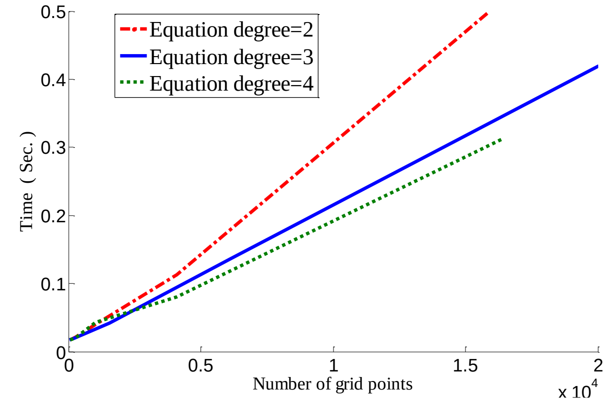

Fig. 8. The solution time versus number of grid points for different equation degree for MCM in case II.

In this study one dimensional (1-D) case is studied, while the extension to two (2-D) and three dimensional (3-D) cases can be easily done. On the other hand, based on new method nature, the convergence and calculation time will be improved when the dimension of problem is increased. The new methods based on iterative techniques give a rapid result with a minimum error to reach the optimum convergence.

TABLE III

NUMBER OF ITERATIONS, NUMBER OF STARTING GRID POINTS, DEGREES OF POLYNOMIAL, ERRORS AND SOLUTION TIME OF EQUATION FOR CASE III.

Methods	***Number of Iterations***							***Number of Starting grid Points (m)***	***Degrees of Polynomial***	***Error***	***Solution Time(sec.)***
Gauss-Seidel	**1868068**							**1025**	**--**	**1e-4**	**258**
DCM	**0**	**1825371**						**2**	**1**	**3.7e-5**	**253**
	2	**2165958**						**3**	**2**	**3.7e-5**	**300**
	22	**2150676**						**4**	**3**	**3.7e-5**	**299**
	40	**1665445**						**5**	**4**	**3.7e-5**	**231**
	66	**398891**						**6**	**5**	**3.7e-5**	**55**
	102	**1623564**						**7**	**6**	**3.7e-5**	**227**
MCM	2	44	164	544	1882	6361	20832	**3**	**1**	**1.1e-4**	**79**
	40	174	514	1559	4464	10918	15626	**5**	**2**	**1.5e-4**	**14**
	102	357	1003	2936	7773	15459	27692	**7**	**3**	**6.6e-5**	**29**
	178	482	1547	4459	10913	15625	40195	**9**	**4**	**1.5e-4**	**13.5**
	281	877	2198	6098	13564	21167	51751	**11**	**5**	**9.5e-5**	**19.4**

IV. CONCLUSION

In this paper two techniques for finite difference solution of partial differential equations are presented. These methods are used to solve three common partial differential equations based on iterative method. The two new methods focus on improvement of faster convergence and accuracy of the solutions. The results show that the MCM has faster speed of convergence than common Gauss-Seidel method and DCM, in other words it leads to reducing the solution time. This outstanding aspect is illustrated for those of equation with combinational response.

REFERENCES

[1] N. Matthew, and O. Sadiku, *Numerical techniques in electromagnetics*: CRC Press Boca Raton, FL, 2001.

[2] V. De Santis, S. Cruciani, M. Feliziani, and M. Okoniewski, “Efficient Low Order Approximation for Surface Impedance Boundary Conditions in Finite-Difference Time-Domain Method,” *IEEE Transactions on Magnetics*, vol. 48, no. 2, pp. 271-274, 2012.

[3] C. T. M. Choi, and S. Shu-Hai, “Simulation of Axon Activation by Electrical Stimulation; Applying Alternating-Direction-Implicit Finite- Difference Time-Domain Method,” *IEEE Transactions on Magnetics*, vol. 48, no. 2, pp. 639-642, 2012.

[4] O. Bottauscio, and A. Manzin, “Spatial Reconstruction of Exchange Field Interactions With a Finite Difference Scheme Based on Unstructured Meshes,” *IEEE Transactions on Magnetics*, vol. 48, no. 11, pp. 3250-3253, 2012.

[5] M. Benhamouche, L. Bernard, L. Pichon, and D. Lesselier, "Three-Dimensional Generalized Finite-Difference Modeling of Electromagnetic Time Reversal: Impact of the Density of Dipoles for the Localization of a Dielectric Obstacle in Free Space," *IEEE Transactions on Magnetics*, vol. 48, no. 2, pp. 359-362, 2012.

[6] C. Abert, G. Selke, B. Kruger, and A. Drews, "A Fast Finite-Difference Method for Micromagnetics Using the Magnetic Scalar Potential," *IEEE Transactions on Magnetics,* vol. 48, no. 3, pp. 1105-1109, 2012.

[7] J. Golde, J. Shertzer, and P. Oxley, "Finite element solution of Laplace's equation for ion-atom chambers," *American Journal of Physics,* vol. 77, pp. 81, 2009.

[8] Z. Jomaa, and C. Macaskill, "The Shortley-Weller embedded finite-difference method for the 3D Poisson equation with mixed boundary conditions," *Journal of Computational Physics,* vol. 229, no. 10, pp. 3675-3690, 2010.

[9] P. Alotto, and F. Freschi, "A Second-Order Cell Method for Poisson's Equation," *IEEE Transactions on Magnetics,* vol. 47, no. 5, pp. 1430-1433, 2011.

[10] B. L. J. Gysen, E. A. Lomonova, J. J. H. Paulides, and A. J. A. Vandenput, "Analytical and Numerical Techniques for Solving Laplace and Poisson Equations in a Tubular Permanent Magnet Actuator: Part II. Schwarz–Christoffel Mapping," *IEEE Transactions on Magnetics*, vol. 44, no. 7, pp. 1761-1767, 2008.

[11] L. Chany, K. Do Wan, P. Sang-Hun, K. Hong-Kyu, I. Chang-Hwan, and J. Hyun-Kyo, "Point Collocation Mesh-Free Method Using FMLSRKM for Solving Axisymmetric Laplace Equation," *IEEE Transactions on Magnetics,* vol. 44, no. 6, pp. 1234-1237, 2008.

[12] H. Torkaman, and F. Karimi, "Measurement variations of insulation resistance/polarization index during utilizing time in HV electrical machines – A survey," *Measurement, Elsevier,* vol. 59, pp. 21-29, 2015.

[13] H. Torkaman, and F. Karimi, "Influence of Ambient and Test Conditions on Insulation Resistance/Polarization Index in HV Electrical Machines–A Survey," *IEEE Transactions on Dielectrics and Electrical Insulation,* vol. 22, no. 1, pp. 241-250, 2015.

[14] M. R. Tavakoli, H. Torkaman, F. Faradjizadeh, and E. Afjei, "Unbalanced Magnetic Force Mitigation and Torque Ripple Reduction in Eccentric SRMs Using Flux Leakages," *Electric Power Components and Systems, Taylor and Francis,* vol. 43, no. 5, pp. 533-542, 2015.

[15] H. Torkaman, "Intern-turn short-circuit fault detection in switched reluctance motor utilizing MCPT test," *International Journal of Applied Electromagnetics and Mechanics,* vol. 46, no. 3, pp. 619-628, 2014.

[16] R. H. Priewald, C. Magele, P. D. Ledger, N. R. Pearson, and J. S. D. Mason, "Fast Magnetic Flux Leakage Signal Inversion for the Reconstruction of Arbitrary Defect Profiles in Steel Using Finite Elements," *IEEE Transactions on Magnetics,* vol. 49, no. 1, pp. 506-516, 2013.

[17] A. Nejadpak, and O. A. Mohammed, "Physics-Based Modeling of Power Converters From Finite Element Electromagnetic Field Computations," *IEEE Transactions on Magnetics,* vol. 49, no. 1, pp. 567-576, 2013.

[18] W. N. Fu, X. Zhang, and S. L. Ho, "A Fast Algorithm for Frequency-Domain Solutions of Electromagnetic Field Computation of Electric Devices Using Time-Domain Finite-Element Method," *IEEE Transactions on Magnetics,* vol. 49, no. 1, pp. 530-535, 2013.

[19] E. Afjei, S. Asoodeh, and A. Dargahi, "Error analysis in finite difference solution of linear and non-linear cylindrical magnetostatic problems," in International Aegean Conference on Electrical Machines and Power Electronics, ACEMP '07., 2007, pp. 232-235.

[20] T. Ohtani, and Y. Kanai, "Optimal Coefficients of the Spatial Finite Difference Operator for the Complex Nonstandard Finite Difference Time-Domain Method," *IEEE Transactions on Magnetics,* vol. 47, no. 5, pp. 1498-1501, 2011.

[21] L. Bernard, R. R. Torrado, and L. Pichon, "Efficient Implementation of the UPML in the Generalized Finite-Difference Time-Domain Method," *IEEE Transactions on Magnetics,* vol. 46, no. 8, pp. 3492-3495, 2010.

[22] B. Van de Wiele, A. Manzin, L. Dupre, F. Olyslager, O. Bottauscio, and M. Chiampi, "Comparison of Finite-Difference and Finite-Element Schemes for Magnetization Processes in 3-D Particles," *IEEE Transactions on Magnetics,* vol. 45, no. 3, pp. 1614-1617, 2009.

[23] E. W. Cheney, and D. R. Kincaid, *Numerical mathematics and computing*: Brooks/Cole Publishing Co. Pacific Grove, CA, USA, 2007.

[24] J. Geiser, "Modified Jacobian Newton Iterative Method: Theory and Applications," *Mathematical Problems in Engineering,* vol. 2009, pp. 1-24, 2009.

[25] I. W. Lee, H. J. Jung, M. C. Kim, and A. R. Robinson, "Modified inverse iteration method using the side condition and the step length," *Journal of Mechanical Science and Technology,* vol. 10, no. 1, pp. 64-71, 1996.

[26] J. Geisera, and C. Kravvaritisb, "Modified Jacobian Newton Iterative Method with Embedded Domain Decomposition Method," *Department of Mathematics, Humboldt University of Berlin,* no. 10, pp. 1-16, 2007.

[27] S. Rajendran, M. R. Raghavan, and M. V. Narasimhan, "Improving the convergence rate of the inverse iteration method," *Computers and Structures,* vol. 55, no. 4, pp. 709-715, 1995.

[28] D. Wan, and S. Turek, "An efficient multigrid-FEM method for the simulation of solid-liquid two phase flows," *Journal of Computational and Applied Mathematics,* vol. 203, no. 2, pp. 561-580, 2007.

[29] H. Yserentant, "Old and new convergence proofs for multigrid methods," *Acta numerica,* vol. 2, pp. 285-326, 2008.

[30] E. Afjei, A. Siadatan, and H. Torkaman, "Magnetic Field Estimation Using Weighted Multi Grid Algorithm," in 6th International Conference of Numerical Analysis and Applied Mathematics (ICNAAM), Greece, 2009, pp. 762-765.